SMOOTH FOURIER MULTIPLIERS ON GROUP VON NEUMANN ALGEBRAS

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Abstract. In this paper, we extend classical methods from harmonic analysis to study smooth Fourier multipliers in the compact dual of arbitrary discrete groups. The main results are a Hörmander-Mihlin type multiplier theorem for finite-dimensional cocycles, Littlewood-Paley inequalities on group algebras and a dimension free L_p estimate for noncommutative Riesz transforms. As a byproduct of our approach, we provide new examples of L_p Fourier multipliers in $\mathbb{R}^n$. The key novelty is to exploit cocycles and cross products in Fourier multiplier theory in conjunction with quantum probability techniques and a noncommutative version of Calderón-Zygmund theory.

Introduction and main results

Convergence of Fourier series and norm estimates for Fourier multipliers are central in harmonic analysis. As far as Calderón-Zygmund methods are involved very few results have been successfully transferred to other noneuclidean topological groups. The impressive work carried out by Müller, Ricci, Stein and their coauthors on nilpotent groups shows that noncommutativity may lead to entirely unexpected results and requires genuinely new ideas, see e.g. [48, 49, 50, 67, 68] and the references therein. Their underlying measure space is usually the group itself or its Lie algebra. Here we follow a different approach, inspired by the ground-breaking results of Haagerup [18] and Cowling/Haagerup [8] on the approximation property and Fourier multipliers on group von Neumann algebras. Indeed, the compact dual of a nonabelian discrete group can only be understood as a quantum group whose underlying space is a group von Neumann algebra, which replaces the former role of the measure space. This general setting is widely accepted and very well understood in noncommutative geometry [5] and operator algebra [37]. Up to isolated contributions [19, 61], the Fourier multiplier theory on these algebras is very much unexplored. This motivates the development of a Calderón-Zygmund theory for von Neumann algebras.

Let G be a discrete group with left regular representation $\lambda : \mathrm{G} \to \mathcal{B}(\ell_2(\mathrm{G}))$ given by $\lambda(g)\delta_h = \delta_{gh}$, where the δ_g's form the unit vector basis of $\ell_2(\mathrm{G})$. Write $\mathcal{L}(\mathrm{G})$ for its group von Neumann algebra, the weak operator closure of the linear span of $\lambda(\mathrm{G})$. Given $f \in \mathcal{L}(\mathrm{G})$, we consider the standard normalized trace $\tau_{\mathrm{G}}(f) = \langle \delta_e, f\delta_e \rangle$ where e denotes the identity element of G. Any such element f has a Fourier series

$$\sum_{g\in \mathrm{G}} \widehat{f}(g)\lambda(g) \quad \text{with} \quad \widehat{f}(g) = \tau_{\mathrm{G}}(f\lambda(g^{-1})) \quad \text{so that} \quad \tau_{\mathrm{G}}(f) = \widehat{f}(e).$$

Let $L_p(\widehat{\mathbb{G}}) = L_p(\mathcal{L}(\mathrm{G}), \tau_\mathrm{G})$ denote the L_p space over the noncommutative measure space $(\mathcal{L}(\mathrm{G}), \tau_\mathrm{G})$ —so called noncommutative L_p spaces— equipped with the norm

$$\|f\|_p = \Big\| \sum\nolimits_g \widehat{f}(g)\lambda(g) \Big\|_p = \Big(\tau_\mathrm{G} \Big[\Big| \sum\nolimits_g \widehat{f}(g)\lambda(g) \Big|^p \Big] \Big)^{\frac{1}{p}}.$$

We invite the reader to check that $L_p(\widehat{\mathbb{G}}) = L_p(\mathbb{T}^n)$ for $\mathrm{G} = \mathbb{Z}^n$, after identifying $\lambda(k)$ with $e^{2\pi i \langle k, \cdot \rangle}$. In the general case, the absolute value and the power p are obtained from functional calculus for this (unbounded) operator on the Hilbert space $\ell_2(\mathrm{G})$. A Fourier multiplier is then given by

$$T_m : \sum\nolimits_g \widehat{f}(g)\, \lambda(g) \mapsto \sum\nolimits_g m_g \, \widehat{f}(g)\, \lambda(g).$$

If $\mathrm{G} = \mathbb{Z}^n$ we find Fourier multipliers on the n-torus. We will say that any smooth function $\widetilde{m} : \mathbb{R}^n \to \mathbb{C}$ is a lifting multiplier for m whenever its restriction to $\mathbb{Z}^n$ coincides with m. According to de Leeuw's restriction theorem [9], the L_p boundedness of T_m follows whenever there exists a lifting multiplier defining an L_p-bounded map in the ambient space $\mathbb{R}^n$. In particular, if $1 < p < \infty$ it suffices to check the Hörmander-Mihlin smoothness condition [21, 47]

$$\big| \partial_\xi^\beta \, \widetilde{m}(\xi) \big| \lesssim |\xi|^{-|\beta|} \quad \text{for all} \quad |\beta| \le \Big[\frac{n}{2} \Big] + 1.$$

In the context of Lie groups we may find similar formulations, where the role of $\mathbb{R}^n$ is replaced by the corresponding Lie algebra. A fundamental goal for us is to give sufficient differentiability conditions for the L_p boundedness of multipliers on the compact dual of discrete groups. Unlike for $\mathbb{Z}^n$, there is no standard differential structure to construct/evaluate lifting multipliers for an arbitrary discrete G. The main novelty in our approach is to identify the right endpoint spaces —intrinsic BMO's over certain semigroups— using a broader interpretation of tangent spaces in terms of length functions and cocycles.

An *affine representation* of G is an orthogonal representation $\alpha : \mathrm{G} \to O(\mathcal{H})$ over a real Hilbert space $\mathcal{H}$ together with a mapping $b : \mathrm{G} \to \mathcal{H}$ satisfying the cocycle law $b(gh) = \alpha_g(b(h)) + b(g)$. In this paper we say that $\psi : \mathrm{G} \to \mathbb{R}_+$ is a *length function* if it vanishes at the identity e, $\psi(g) = \psi(g^{-1})$ and is conditionally negative which means that $\sum_g \beta_g = 0 \Rightarrow \sum_{g,h} \overline{\beta}_g \beta_h \psi(g^{-1}h) \le 0$. According to Schoenberg's theorem [72], any length function $\psi : \mathrm{G} \to \mathbb{R}_+$ determines an affine representation $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$ and viceversa. Hörmander-Mihlin and de Leeuw classical theorems are formulated in terms of the standard cocycle given by the heat semigroup. We propose the Hilbert spaces $\mathcal{H}_\psi$ as cocycle substitutes of the Lie algebra. Here is a fairly simple formulation —see also Theorems B and 2.4— of our cocycle form of Hörmander-Mihlin theorem, valid for group von Neumann algebras.

Theorem A. *Let* G *be a discrete group and*

$$T_m : \sum\nolimits_g \widehat{f}(g)\lambda(g) \mapsto \sum\nolimits_g m_g \widehat{f}(g)\lambda(g).$$

Let ψ be a length function with $\dim \mathcal{H}_\psi = n < \infty$ and $\widetilde{m} : \mathcal{H}_\psi \to \mathbb{C}$ such that

a) *$\widetilde{m}$ is a ψ-lifting of m: $m_g = \widetilde{m}(b_\psi(g))$,*

b) $\big| \partial_\xi^\beta \widetilde{m}(\xi) \big| \lesssim \min \Big\{ |\xi|^{-|\beta| + \varepsilon}, |\xi|^{-|\beta| - \varepsilon} \Big\}$ *for $|\beta| \le [\frac{n}{2}] + 1$ and some $\varepsilon > 0$.*

Then, $T_m : L_p(\widehat{\mathbb{G}}) \to L_p(\widehat{\mathbb{G}})$ is a completely bounded multiplier for all $1 < p < \infty$.

Completely bounded (cb) means that $T_m \otimes id$ is a multiplier on $L_p(\widehat{\mathbb{G} \times \mathbb{H}})$ for every discrete group H. The additional ε is a prize we pay for noncommutativity which can be removed under alternative assumptions, like

i) G is abelian,
ii) $b_\psi(\mathrm{G})$ is a lattice in $\mathbb{R}^n$,
iii) $\alpha_\psi(\mathrm{G})$ is a finite subgroup of $O(n)$,
iv) The multiplier is ψ-radial, i.e. $m_g = h(\psi(g))$.

Theorem A is a cocycle extension of the Mihlin multiplier theorem, more than merely a noncommutative form of it. Indeed, it provides new results even for finite or Euclidean groups. For instance, we may find low dimensional injective cocycles for finite groups of large cardinality, like $\mathbb{Z}_n$ or the symmetric permutation groups S_n, where we find injective cocycles with $\dim \mathcal{H}_\psi = 2 << n$ and $\dim \mathcal{H}_\psi = n << n!$ respectively, see [27, Section 5.2]. To the best of our knowledge, there are no similar results for finite groups. In the context of $\mathbb{R}^n$, Theorem A shows that

$$T_m f(x) = \int_{\mathbb{R}^n} \widetilde{m}(b(\xi)) \, \widehat{f}(\xi) \, e^{2\pi i \langle x, \xi \rangle} \, d\xi$$

is $L_p(\mathbb{R}^n)$-bounded for any cocycle $b : \mathbb{R}^n \to \mathbb{R}^d$ with $\widetilde{m}$ Mihlin-smooth of degree $[\frac{d}{2}] + 1$. By picking suitable cocycles, this establishes a unified approach through de Leeuw's restriction/periodization theorems [9]. Hörmander-Mihlin theorem also corresponds to the trivial cocycle on $\mathbb{R}^n$, and new $L_\infty \to \mathrm{BMO}$ estimates will be given. In fact, other cocycles provide a large family of L_p multipliers in $\mathbb{R}^n$ —also in $\mathbb{T}^n$— which are apparently new. As an illustration, take

$$b(\xi) = (\cos 2\pi\alpha\xi - 1, \sin 2\pi\alpha\xi, \cos 2\pi\beta\xi - 1, \sin 2\pi\beta\xi)$$

for some $\alpha, \beta \in \mathbb{R}_+$. Theorem A shows that the restriction of a Mihlin multiplier in $\mathbb{R}^4$ to this *donut helix* will be an L_p multiplier on $\mathbb{R}$. It is useful to compare it with de Leeuw's periodization theorem for compactly supported multipliers. The main difference here is the irregularity obtained from choosing α/β irrational, leading to a geodesic flow with dense orbit. Hence, m oscillates infinitely often with no periodic pattern. Taking for instance $\widetilde{m}(\zeta) = |\zeta|^{2\gamma}$ with $0 < \gamma < \frac{1}{2}$ and smoothly truncated outside $\mathrm{B}_3(0)$, Theorem A shows that $(\sin^2(\alpha\xi) + \sin^2(\beta\xi))^\gamma$ is an L_p-multiplier in $\mathbb{R}$. These examples are certainly less standard in harmonic analysis. With hindsight, they can be obtained via a clever combination of classical results, we invite the reader to try! However, it seems fair to say that noncommutative inspiration was required to discover such a general statement, see [55, 58] for related results.

The presence of $\varepsilon > 0$ in Theorem A excludes some central examples, like the *ψ-directional Riesz transforms* which —recalling that $\psi(g) = \langle b_\psi(g), b_\psi(g) \rangle_\psi$— are naturally defined for $\eta \in \mathcal{H}_\psi$ as

$$R_\eta \Big(\sum\nolimits_g \widehat{f}(g) \lambda(g) \Big) = -i \sum\nolimits_g \frac{\langle b_\psi(g), \eta \rangle_\psi}{\sqrt{\psi(g)}} \widehat{f}(g) \lambda(g).$$

These multipliers are covered by Theorem B below, which exploits the semigroup $S_{\psi,t} : \lambda(g) \mapsto e^{-t\psi(g)} \lambda(g)$ and the BMO space constructed with it via the norm $\|f\|_{\mathrm{BMO}_{\mathcal{S}_\psi}} = \max\{ \|f\|_{\mathrm{BMO}^c_{\mathcal{S}_\psi}}, \|f^*\|_{\mathrm{BMO}^c_{\mathcal{S}_\psi}} \}$, where

$$\|f\|_{\mathrm{BMO}^c_{\mathcal{S}_\psi}} = \sup_{t>0} \Big\| \Big(S_{\psi,t}|f|^2 - |S_{\psi,t} f|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathcal{L}(\mathrm{G})} \quad \text{with} \quad |f|^2 = f^* f.$$

Theorem B. *Given* (G, ψ) *and*

$$T_m : \sum\nolimits_g \widehat{f}(g)\lambda(g) \mapsto \sum\nolimits_g m_g \widehat{f}(g)\lambda(g)$$

as above, let $\widetilde{m} : \mathcal{H}_\psi \to \mathbb{C}$ *be a lifting multiplier for* $m = \widetilde{m} \circ b_\psi$ *such that*

i) *L_2-row/column condition*

$$\big\|\widetilde{m}\big\|_{schur} \ = \inf_{\substack{\widetilde{m}(\alpha_{\psi,g}(\xi)) = \langle A_\xi, B_g\rangle_{\mathcal{K}} \\ \mathcal{K} \ \mathrm{Hilbert}}} \Big(\sup_{\xi \in \mathcal{H}_\psi} \|A_\xi\|_{\mathcal{K}} \sup_{g \in \mathrm{G}} \|B_g\|_{\mathcal{K}} \Big) \ < \ \infty.$$

ii) *Hörmander-Mihlin smoothness*

$$\widetilde{m} \in \mathcal{C}^{n+2}(\mathbb{R}^n \setminus \{0\}) \quad and \quad \big|\partial_\xi^\beta \widetilde{m}(\xi)\big| \ \le \ c_n |\xi|^{-|\beta|} \quad for\ all \quad |\beta| \le n+2.$$

Then, we find $T_m : \mathcal{L}(\mathrm{G}) \xrightarrow{cb} \mathrm{BMO}_{\mathcal{S}_\psi}$ *and* $T_m : L_p(\widehat{\mathbb{G}}) \xrightarrow{cb} L_p(\widehat{\mathbb{G}})$ *for* $1 < p < \infty$*.*

Theorems A and B can be used either to construct Fourier multipliers or to test the L_p boundedness of a fixed multiplier. The real challenge in the latter case is to find the right length/cocycle b_ψ and the lifting $\widetilde{m}$ such that $m_g = \widetilde{m}(b_\psi(g))$. This is exactly the topic of Fefferman's *smooth interpolation of data* [13, 14, 15] relative to the set $b_\psi(\mathrm{G})$. If $\Delta_\psi = \inf_{g \neq h} \|b(g) - b(h)\|_{\mathcal{H}_\psi}^2 > 0$, we say that the cocycle $b_\psi : \mathrm{G} \to \mathcal{H}_\psi$ is well-separated. The Hörmander-Mihlin theorem for the n-torus corresponds to the standard cocycle $\mathbb{Z}^n \subset \mathbb{R}^n$ with the trivial action α. Up to changes of basis, it is the only finite-dimensional, injective, well-separated cocycle of $\mathbb{Z}^n$, see Lemma 5.14. Accordingly, we call $b_\psi : \mathrm{G} \to \mathcal{H}_\psi$ standard if it is injective and well-separated. By a classical theorem of Bieberbach [2], G admits a standard finite-dimensional cocycle if and only if it is virtually abelian. This excludes for instance discrete groups with Kazhdan's property (T). The novelty in our approach is to allow for *clustering* in the set $b_\psi(\mathrm{G})$ and thus go beyond the class of virtually abelian groups. Of course, our hypotheses lead to look for disperse clouds $b_\psi(\mathrm{G})$ living in low dimensional spaces $\mathcal{H}_\psi$. We refer to Paragraph 5.7 for a description of the intriguing interplay between these "competing" requirements.

Our methods lead to related Littlewood-Paley estimates. Consider a sequence of functions $(h_m)_{m \ge 1}$ in $\mathcal{C}^{[\frac{n}{2}]+1}(\mathbb{R}_+ \setminus \{0\})$ such that $\sum_m |\frac{d^k}{d\xi^k} h_m(\xi)|^2 \ \le \ c_n \, |\xi|^{-2k}$ for all $k \le [\frac{n}{2}] + 1$. Given a length function $\psi : \mathrm{G} \to \mathbb{R}_+$, define

$$T_m f = \sum\nolimits_g h_m(\psi(g)) \widehat{f}(g) \lambda(g)$$

for $f = \sum_g \widehat{f}(g)\lambda(g)$. Then, the following square function L_p inequalities hold.

Theorem C. *If* $1 < p < \infty$*, we have*

$$\Big\| \sum_{m=1}^\infty T_m f \otimes \delta_m \Big\|_{L_p(\widehat{\mathbb{G}}; \ell_{rc}^2)} \ \le_{cb} \ c_p \, \|f\|_{L_p(\widehat{\mathbb{G}})}.$$

Additionally, we have $L_\infty \to \mathrm{BMO}$ *type inequalities and*

$$\sum_{m=1}^\infty |h_m(\xi)|^2 = 1 \quad \Rightarrow \quad \|f\|_{L_p(\widehat{\mathbb{G}})} \ \le_{cb} \ c_p \Big\| \sum_{m=1}^\infty T_m f \otimes \delta_m \Big\|_{L_p(\widehat{\mathbb{G}}; \ell_{rc}^2)}.$$

Theorem C has been a crucial tool for the main result in [58], see Theorem 3.3 for a more precise formulation. A noncommutative Calderón-Zygmund theory requires to find substitutes for the interplay metric/measure in commutative spaces. Theorems A, B and C emerge from CZO's on von Neumann algebras $\mathcal{R} \rtimes \mathrm{G}$, where G acts on a von Neumann algebra $\mathcal{R}$ which factors as a tensor product of $\mathbb{R}^n$ with its Lebesgue measure and any other noncommutative measure space $(\mathcal{M}, \tau)$. The key link with our main results comes from the *intertwining identities*

$$\pi_\psi \circ S_{\psi,t} = \big(S_t \rtimes id_{\mathcal{L}(\mathrm{G})}\big) \circ \pi_\psi \quad \text{and} \quad \pi_\psi \circ T_m = \big(T_{\widetilde{m}} \rtimes id_{\mathcal{L}(\mathrm{G})}\big) \circ \pi_\psi,$$

where $(S_t)_{t \ge 0}$ denotes the heat semigroup on $\mathcal{H}_\psi$ and $\pi_\psi : \mathcal{L}(\mathrm{G}) \to \mathcal{L}(\mathcal{H}_\psi) \rtimes_{\alpha_\psi} \mathrm{G}$ is the $*$-homomorphism $\lambda(g) \mapsto \exp(2\pi i \langle b_\psi(g), \cdot \rangle_\psi) \rtimes \lambda(g)$. The first intertwining identity yields an embedding $\mathrm{BMO}_{\mathcal{S}_\psi} \to \mathrm{BMO}_{\mathcal{S}_\rtimes}$ with $S_{\rtimes,t} = S_t \rtimes id_{\mathcal{L}(\mathrm{G})}$. This explains our interest on BMO spaces over semigroups of cp maps. In the classical theory, we find BMO spaces associated to a metric or a martingale filtration. Duong and Yan [10, 11] —see also [1, 20]— extended it to certain semigroups of positive maps, but still imposing the existence of a metric in the underlying space, something that a priori we do not have at our disposal. Interpolation results with L_p spaces [26] are deduced from the theory of noncommutative martingales with continuous index set [31] and a theory of Markov dilations with continuous path [32].

Given a semigroup of Fourier multipliers $\mathcal{S}_\psi = (S_{\psi,t})_{t \ge 0}$ determined by a length function $\psi : \mathrm{G} \to \mathbb{R}_+$ as above, its infinitesimal generator $A_\psi(\lambda(g)) = \psi(g)\lambda(g)$ yields the gradient —Meyer's "carré du champs"— given by

$$\Gamma_\psi(f,f) = \frac{1}{2}\big(A_\psi(f^*)f + f^*A_\psi(f) - A_\psi(f^*f)\big).$$

Our next result establishes a new *dimension free estimate* for Riesz transforms.

Theorem D. *If $2 \le p < \infty$, we find*

$$\big\|\Gamma_\psi(f,f)^{\frac12}\big\|_p + \big\|\Gamma_\psi(f^*,f^*)^{\frac12}\big\|_p \sim \big\|A_\psi^{\frac12} f\big\|_p.$$

Theorem A shows some impact of cohomology on Fourier multiplier theory. Tools from classical harmonic analysis impose the condition $\dim \mathcal{H}_\psi < \infty$. However, many interesting cocycles are constructed on infinite-dimensional Hilbert spaces. In fact certain exotic groups like the Tarski monster or some Burnside groups, do not admit finite-dimensional cocycles at all. Fortunately, this is not the end of Fourier multiplier theory. Theorem D is a noncommutative form of Meyer's dimension free estimates for the Riesz transform, Theorem 4.6 also includes the correct formulation for $1 < p \le 2$. Our proof is based on Pisier's method, which has been exploited in a similar context by Lust-Piquard [40, 41, 42, 43, 44]. The new ingredient is a noncommutative Khintchine type inequality of independent interest, which leads to a sum of spaces for $p < 2$ and an intersection for $p > 2$. As an application the ψ-directional Riesz transforms are still bounded on L_p for infinite-dimensional cocycles, see Corollary 4.7. Theorem D also provides apparently new multipliers in $\mathbb{R}^n$, other examples will be given for noncommutative tori and free group algebras. The main result in [25] included the lower estimate in Theorem D for $p \ge 2$.

Let $\widehat{\mathbb{R}}^n_{\mathrm{disc}}$ denote the Bohr compactification of $\mathbb{R}^n$. Since $\psi(g) = \langle b_\psi(g), b_\psi(g) \rangle_\psi$ a function $m : \mathrm{G} \to \mathbb{C}$ is called ψ-radial if $m_g = h(\psi(g))$ for some $h : \mathbb{R}_+ \to \mathbb{C}$, so that we find a lifting $\widetilde{m}$ which is radial on $\mathcal{H}_\psi$. We will use the little Grothendieck theorem [17] for the following characterization of ψ-radial Fourier multipliers.

Theorem E. *If* $h : \mathbb{R}_+ \to \mathbb{C}$, TFAE

i) $T_{h \circ | \, |^2} : L_\infty(\mathbb{R}^d) \to \mathrm{BMO}_\mathcal{S}(\mathbb{R}^d)$ *bounded,*
ii) $T_{h \circ | \, |^2} : L_\infty(\widehat{\mathbb{R}}^d_{\mathrm{disc}}) \to \mathrm{BMO}_\mathcal{S}(\widehat{\mathbb{R}}^d_{\mathrm{disc}})$ *bounded,*
iii) $T_{h \circ \psi} : \mathcal{L}(\mathrm{G}) \to \mathrm{BMO}_{\mathcal{S}_\psi}$ *bounded for all* G *discrete with* $\dim \mathcal{H}_\psi = d$,

where $\mathcal{S}$ *denotes the heat semigroup. Moreover,* ii) $\Leftrightarrow$ iii) *still holds for* $d = \infty$.

Since $A_\psi(\lambda(g)) = \psi(g)\lambda(g)$ generates $\mathcal{S}_\psi$, radial multipliers are of the form $h(A_\psi)$, already considered by McIntosh's H_∞-calculus for analytic h. Theorem A imposes considerably weaker conditions and Theorem E provides new Fourier multipliers even for infinite-dimensional cocycles. The imaginary powers $\psi(g)^{is}$ and other examples of Laplace transforms are included, see [26]. As an illustration in $\mathrm{G} = \mathbb{R}^n$, the length functions $\psi(\xi) = 1 - \int_{\mathbb{R}^n} f_0(x) f_0(x-\xi) \, dx$ with $\|f_0\|_2 = 1$ or $\psi(\xi) = \int_\Omega |\sum_j \xi_j f_j| \, d\mu$ with $f_j \in L_1(\Omega, \mu)$ come from infinite-dimensional cocycles and taking $f_0 = \chi_\Sigma$, we obtain highly irregular ψ's. These $m_\xi = \psi(\xi)^{is}$ are just exotic forms of Stein's imaginary powers and L_p-boundedness for $1 < p < \infty$ is guaranteed. The main novelty from our method is that we may identify the endpoint estimates for T_m associated to $m_\xi = \psi(\xi)^{is}$, so that $T_m : L_\infty(\mathbb{R}^n) \to \mathrm{BMO}_{\mathcal{S}_\psi}$.

At the end of the paper, we will illustrate our main results with applications on noncommutative tori and the free group algebra. We will also construct in Corollary 5.12 new examples of Rieffel's quantum metric spaces for the compact dual of virtually abelian discrete groups.

The paper essentially follows the order established in this Introduction. Our approach requires some background on von Neumann algebras, noncommutative L_p and Hardy spaces, as well as some operator space terminology. Standard references on operator algebra are [37, 75]. We refer to [54, Section 1] for a brief review of the results from noncommutative integration needed for this paper. A more in depth discussion is given in Pisier/Xu's survey [66]. The p-norms of row/column square functions and corresponding Hardy spaces appear in [24, 62, 65]. Two excellent books on operator space theory are Effros/Ruan and Pisier monographs [12, 64]. A noncommutative measure space will denote a pair $(\mathcal{M}, \tau)$ formed by a semifinite von Neumann algebra $\mathcal{M}$ and a normal semifinite faithful trace τ on it[†].

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[†]This paper is substantially different from the more preliminary version [27]. Namely, despite a considerable size reduction, some new and crucial dimension free estimates have been added in Theorem D and Section 4. We have also reduced the Mihlin regularity in Theorem A to the expected degree $|\beta| \leq [\frac{n}{2}] + 1$.

1. Twisted CZO's

We first study the $L_\infty \to \mathrm{BMO}$ boundedness of semidirect product extensions of semicommutative CZO's. Our results are of independent interest, regarded as a first step through Calderón-Zygmund theory on fully noncommutative von Neumann algebras, see [28, 29] for related results.

1.1. **Crossed products.** Given a discrete group G with left regular representation $\lambda : \mathrm{G} \to \mathcal{B}(\ell_2(\mathrm{G}))$, let $\mathcal{L}(\mathrm{G})$ denote its group von Neumann algebra and $L_p(\widehat{\mathbb{G}})$ the associated noncommutative L_p space, as defined in the Introduction. Note that for G abelian we get the L_p space on the dual group equipped with its normalized Haar measure. We will sometimes keep the terminology $\mathcal{L}(\mathrm{G})$ for $p = \infty$. Given another noncommutative measure space $(\mathcal{M}, \tau)$ with $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, assume that there exists a trace preserving action $\alpha : \mathrm{G} \to \mathrm{Aut}(\mathcal{M})$. Define the crossed product algebra $\mathcal{M} \rtimes_\alpha \mathrm{G}$ as the weak operator closure in $\mathcal{B}(\ell_2(\mathrm{G}; \mathcal{H}))$ of the $*$-algebra generated by $\mathbf{1}_\mathcal{M} \otimes \lambda(\mathrm{G})$ and $\rho(\mathcal{M})$, where the $*$-representation $\rho : \mathcal{M} \to \mathcal{B}(\ell_2(\mathrm{G}; \mathcal{H}))$ is given by $\rho(f) = \sum_{h \in \mathrm{G}} \alpha_{h^{-1}}(f) \otimes e_{h,h}$, with $e_{g,h}$ the matrix units for $\ell_2(\mathrm{G})$. A generic element of $\mathcal{M} \rtimes_\alpha \mathrm{G}$ can be formally written as $\sum_{g \in \mathrm{G}} f_g \rtimes_\alpha \lambda(g)$, where each $f_g \in \mathcal{M}$. Playing with the representations λ and ρ, it is clear that $\mathcal{M} \rtimes \mathrm{G}$ sits in $\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathrm{G}))$

$$\begin{aligned}
\sum\nolimits_g f_g \rtimes \lambda(g) &= \sum\nolimits_g \rho(f_g)\lambda(g) \\
&= \sum\nolimits_{g,h,h'} \big(\alpha_{h^{-1}}(f_g) \otimes e_{h,h}\big)\big(\mathbf{1}_\mathcal{M} \otimes e_{gh',h'}\big) \\
&= \sum\nolimits_{g,h} \alpha_{h^{-1}}(f_g) \otimes e_{h,g^{-1}h} \ = \ \sum\nolimits_{g,h} \alpha_{g^{-1}}(f_{gh^{-1}}) \otimes e_{g,h}.
\end{aligned}$$

Similar computations lead to

- $(f \cdot \lambda(g))^* = \alpha_{g^{-1}}(f^*) \cdot \lambda(g^{-1})$,
- $(f \cdot \lambda(g))(f' \cdot \lambda(g')) = f\alpha_g(f') \cdot \lambda(gg')$,
- $\tau \rtimes \tau_\mathrm{G}(f \cdot \lambda(g)) = \tau \otimes \tau_\mathrm{G}(f \otimes \lambda(g)) = \delta_{g=e}\tau(f)$.

Since α will be fixed, we relax the terminology and write $\sum_g f_g\lambda(g) \in \mathcal{M} \rtimes \mathrm{G}$ for generic elements in the cross product. We say that a semigroup $\mathcal{S} = (S_t)_{t \ge 0}$ on $(\mathcal{M}, \tau)$ is G-*equivariant* if $\alpha_g S_t = S_t \alpha_g$ for $(t, g) \in \mathbb{R}_+ \times \mathrm{G}$. Let $\mathcal{S}_\rtimes = (S_t \rtimes id_\mathrm{G})_{t \ge 0}$ and $\mathcal{S}_\otimes = (S_t \otimes id_{\mathcal{B}(\ell_2(\mathrm{G}))})_{t \ge 0}$ denote the cross/tensor product amplification of our semigroup on $\mathcal{M} \rtimes \mathrm{G}$ and $\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathrm{G}))$ respectively.

Lemma 1.1. *If $\mathcal{S}$ is* G*-equivariant, the inclusion map*

$$j : \sum_{g \in \mathrm{G}} f_g \lambda(g) \mapsto \sum_{g,h \in \mathrm{G}} \alpha_{g^{-1}}(f_{gh^{-1}}) \otimes e_{g,h}$$

extends to a complete isometry $\mathrm{BMO}_{\mathcal{S}_\rtimes}(\mathcal{M} \rtimes \mathrm{G}) \to \mathrm{BMO}_{\mathcal{S}_\otimes}(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathrm{G})))$.

Proof. We just prove the column case, the row case is similar. It follows from the definition of $\mathcal{M} \rtimes \mathrm{G}$ that $j : \mathcal{M} \rtimes \mathrm{G} \to \mathcal{M} \bar{\otimes} \mathcal{B}(\ell(\mathrm{G}))$ is a cb-isometry. Letting $f = \sum_g f_g \lambda(g)$ and since $\mathcal{S}$ is G-equivariant it can be checked that

$$S_{\rtimes,t}|f|^2 - |S_{\rtimes,t} f|^2 = \sum_{g,h \in \mathrm{G}} \alpha_{g^{-1}}\big(S_t(f_g^* f_h) - S_t(f_g^*) S_t(f_h)\big)\lambda(g^{-1}h).$$

Then, simple algebraic calculations give rise to

$$j\Big(S_{\rtimes,t}|f|^2 - |S_{\rtimes,t}f|^2\Big) = S_{\otimes,t}|j(f)|^2 - |S_{\otimes,t}j(f)|^2.$$

The same identities hold after matrix amplification and we obtain the assertion. □

1.2. **Equivariant CZO's.** We now show how G-equivariant, $L_\infty \to \mathrm{BMO}$ bounded normal maps extend to crossed products. Given a discrete group G and a pair of noncommutative measure spaces $(\mathcal{M}_j, \tau_j)$ for $j = 1, 2$, assume that $\mathrm{G} \curvearrowright \mathcal{M}_j$ by trace preserving actions α_j. Let $\mathcal{S}_2 = (S_{2,t})_{t\ge 0}$ denote a G-equivariant diffusion semigroup on $(\mathcal{M}_2, \tau_2)$. Now consider a normal map $T : \mathcal{A}_1 \to \mathrm{BMO}_{\mathcal{S}_2}$ defined on a weakly dense $*$-subalgebra $\mathcal{A}_1$ of $\mathcal{M}_1$ such that $T(\mathcal{A}_1) \subset \mathcal{M}_2$. Then we say that T is G-equivariant if $\alpha_{2,g}Tf = T\alpha_{1,g}f$ for all $g \in \mathrm{G}$ and all $f \in \mathcal{A}_1$. Note that we may not have $\alpha_{1g}(\mathcal{A}_1) \subset \mathcal{A}_1$ for all $g \in \mathrm{G}$, so that the right hand side is a priori not well-defined. However, the normality of T provides a bounded extension $T : \mathcal{M}_1 \to \mathrm{BMO}_{\mathcal{S}_2}$. As we shall see below in this paper, radial Fourier multipliers are central examples of G-equivariant CZO's.

Lemma 1.2. *If $T : \mathcal{A}_1 \xrightarrow{cb} \mathrm{BMO}_{\mathcal{S}_2}$ is G-equivariant, we find that*

$$T \rtimes id_{\mathrm{G}} : \mathcal{M}_1 \rtimes \mathrm{G} \to \mathrm{BMO}_{\mathcal{S}_{2\rtimes}}(\mathcal{M}_2 \rtimes \mathrm{G}) \quad \textit{is also completely bounded.}$$

The same conclusion holds when T is only bounded, but the $\mathcal{M}_j$'s are commutative.

Proof. Given $g \in \mathrm{G}$ and $f_g \in \mathcal{A}_1$, we have

$$j_2\big(T \rtimes id_{\mathrm{G}}(f_g\lambda(g))\big) = \sum\nolimits_h T(\alpha_{1,g^{-1}}(f_{gh^{-1}})) \otimes e_{g,h} = T \otimes id_{\mathcal{B}(\ell_2(\mathrm{G}))}\big(j_1(f_g\lambda(g))\big)$$

by G-equivariance of T. By linearity and normality we find that $T \rtimes id_{\mathrm{G}} = j_2^{-1} \circ (T \otimes id_{\mathcal{B}(\ell_2(\mathrm{G}))}) \circ j_1$. On the other hand, since T is cb-bounded we find a normal cb-map $T \otimes id_{\mathcal{B}(\ell_2(\mathrm{G}))} : \mathcal{A}_1 \bar{\otimes} \mathcal{B}(\ell_2(\mathrm{G})) \to \mathrm{BMO}_{\mathcal{S}_{2\otimes}}(\mathcal{M}_2 \bar{\otimes} \mathcal{B}(\ell_2(\mathrm{G})))$, whose normal extension to $\mathcal{M}_1 \bar{\otimes} \mathcal{B}(\ell_2(\mathrm{G}))$ remains cb-bounded. Then, according to Lemma 1.1 and the observations above, $T \rtimes id_{\mathrm{G}}$ is a cb-map $T \rtimes id_{\mathrm{G}} : \mathcal{M}_1 \rtimes \mathrm{G} \to \mathrm{BMO}_{\mathcal{S}_{2\rtimes}}(\mathcal{M}_2 \rtimes \mathrm{G})$ and the first assertion is proved. For the second one, we may assume that $(\mathcal{M}_j, \tau_j)$ is of the form $L_\infty(\Omega_j, \mu_j)$. According to the first part of the statement it suffices to see that any bounded map $T : L_\infty(\Omega_1) \to \mathrm{BMO}_{\mathcal{S}_2}(\Omega_2)$ is indeed cb-bounded. Our argument is row/column symmetric and we just consider the column case. Given a matrix-valued function $f = (f_{ij}) : \Omega_1 \to M_m$, we have

$$\begin{aligned}
\big\|Tf\big\|_{M_m(\mathrm{BMO}^c_{\mathcal{S}_2}(\Omega_2))} &= \sup_{t\ge 0} \Big\| \Big(S_{2,t}|Tf|^2 - |S_{2,t}Tf|^2\Big)^{\frac12} \Big\|_{L_\infty(\Omega_2;M_m)} \\
&= \sup_{\substack{t\ge 0 \\ \|\xi\|_{\ell_2(m)}\le 1}} \ \operatorname*{ess\,sup}_{w\in\Omega_2} \Big\langle \xi, \Big[S_{2,t}|Tf|^2 - |S_{2,t}Tf|^2\Big](w)\xi \Big\rangle_{\ell_2}^{\frac12} \\
&\sim \Big\langle \xi, \Big(-\!\!\!\!\!\!\int_\Sigma \Big[S_{2,t}|Tf|^2 - |S_{2,t}Tf|^2\Big](w)\, d\mu_2(w)\Big)\xi \Big\rangle_{\ell_2(m)}^{\frac12} \\
&= \Big\langle \xi, \Big(\int_{\Omega_2} \Big[S_{2,t}|Tf|^2 - |S_{2,t}Tf|^2\Big](w)\, d\mu_\Sigma(w)\Big)\xi \Big\rangle_{\ell_2(m)}^{\frac12}
\end{aligned}$$

for some fixed ξ, t, some set $\Sigma \in \Omega_2$ of finite positive measure and where μ_Σ stands for the conditional probability measure $\mu_\Sigma(A) = \mu_2(A \cap \Sigma)/\mu_2(\Sigma)$. On the other hand, $S_{2,t}|Tf|^2 - |S_{2,t}Tf|^2 = \langle z, z\rangle$ with $z = Tf \otimes \mathbf{1}_{\Omega_2} - \mathbf{1}_{\Omega_2} \otimes S_{2,t}Tf$ and where

the Hilbert module bracket is given by $\langle a \otimes b, a' \otimes b' \rangle = b^* S_{2,t}(a^*a')b'$. According to the characterization of Hilbert modules in [59], we may find a weak-$*$ continuous right $L_\infty(\Omega_2, \mu_\Sigma)$-module map $u : L_\infty(\Omega_2, \mu_\Sigma) \bar{\otimes}_{S_{2,t}} L_\infty(\Omega_2, \mu_\Sigma) \to L_\infty(\Omega_2, \mu_\Sigma) \bar{\otimes} \mathcal{H}_c$ satisfying $\langle z, z \rangle = |u(z)|^2$. If we define $v(f) = u(Tf \otimes \mathbf{1}_{\Omega_2} - \mathbf{1}_{\Omega_2} \otimes S_{2,t}Tf)$, then we have

$$\begin{aligned} &\big\| Tf \big\|_{M_m(\mathrm{BMO}^c_{\mathcal{S}_2}(\Omega_2))} \\ &\qquad \sim \ \Big\langle \xi, \Big(\int_{\Omega_2} |v(f)(w)|^2 \, d\mu_\Sigma(w) \Big) \xi \Big\rangle_{\ell_2(m)}^{\frac12} \ \le \ \big\| v(f) \big\|_{M_m(L_2^c(\Omega_2, \mu_\Sigma; \mathcal{H}))}. \end{aligned}$$

Therefore, we have reduced the problem to show that $v : L_\infty(\Omega_1) \to L_2^c(\Omega_2, \mu_\Sigma; \mathcal{H})$ is a cb-map. Note that v is normal. Assume for a moment that v is bounded when regarded as a Banach space operator. By the little Grothendieck inequality, this means that v is absolutely 2-summing so that we can find a factorization $v = w \circ j_\xi$ where $j_\xi : f \in L_\infty(\Omega_1, \mu_1) \mapsto f\xi \in L_2(\Omega_1, \mu_1)$ with $\int_{\Omega_1} |\xi|^2 \, d\mu_1 = 1$ and we have $\|w\| \le \frac{2}{\sqrt{\pi}} \|v\|$. This immediately gives that

$$\|v\|_{cb} \le \big\| w : L_2^c(\Omega_1, \mu_1) \to L_2^c(\Omega_2, \mu_\Sigma; \mathcal{H}) \big\|_{cb} \big\| j_\xi : L_\infty(\Omega_1, \mu_1) \to L_2^c(\Omega_1, \mu_1) \big\|_{cb}$$

and yields $\|v\|_{cb} \le \|w\| \le 2/\sqrt{\pi} \|v\|$, because j_ξ is a complete contraction and column Hilbert spaces are homogeneous operator spaces, see e.g. [64]. Thus, we just need to compute the Banach space norm of v. However, applying again the properties of the right module map u, we obtain for $z = Tf \otimes \mathbf{1}_{\Omega_2} - \mathbf{1}_{\Omega_2} \otimes S_{2,t}Tf$

$$\begin{aligned} \|v(f)\|_{L_2(\Omega_2, \mu_\Sigma; \mathcal{H})} \ &\le \ \|v(f)\|_{L_\infty(\Omega_2, \mu_\Sigma; \mathcal{H})} \\ &= \ \|v(f)\|_{L_\infty(\Omega_2, \mu_\Sigma) \bar{\otimes} \mathcal{H}_c} \ = \ \|u(z)^* u(z)\|_{L_\infty(\Omega_2, \mu_\Sigma)}^{\frac12} \\ &= \ \big\| S_{2,t}|Tf|^2 - |S_{2,t}Tf|^2 \big\|_{L_\infty(\Omega_2, \mu_\Sigma)}^{\frac12} \ = \ \|Tf\|_{\mathrm{BMO}^c_{\mathcal{S}_2}(\Omega_2)}. \end{aligned}$$

Hence, $\|v : L_\infty(\Omega_1, \mu_1) \to L_2(\Omega_2, \mu_\Sigma; \mathcal{H})\| \le \|T : L_\infty(\Omega_1, \mu_1) \to \mathrm{BMO}^c_{\mathcal{S}_2}(\Omega_2)\|$. $\square$

1.3. **Semicommutative CZO's.** Given noncommutative measure spaces $(\mathcal{M}_j, \tau_j)$ for $j = 1, 2$, we will write $(\mathcal{R}_j, \varphi_j)$ to denote the von Neumann algebra generated by essentially bounded functions $f : \mathbb{R}^n \to \mathcal{M}_j$ which comes equipped with the trace $\varphi_j(f) = \int_{\mathbb{R}^n} \tau_j(f(y)) \, dy$. In other words, we have $\mathcal{R}_j = L_\infty(\mathbb{R}^n) \bar{\otimes} \mathcal{M}_j$ and our goal is to analyze conditions for the $L_\infty(\mathcal{R}_1) \to \mathrm{BMO}_{\mathcal{R}_2}$ boundedness of CZO's formally given by

$$Tf(x) = \int_{\mathbb{R}^n} k(x, y) \, (f(y)) \, dy$$

with $x \notin \mathrm{supp}_{\mathbb{R}^n} f$ and $k(x, y)$ a linear map from τ_1-measurable to τ_2-measurable operators. Note that $L_p(\mathcal{R}_j) = L_p(\mathbb{R}^n; L_p(\mathcal{M}_j))$, so that this framework does not fall in the vector-valued theory because we take values in different Banach spaces for different values of p, see [54] for further explanations. We also recall that $\|f\|_{\mathrm{BMO}_\mathcal{R}}$ equals

$$\max \big\{ \|f\|_{\mathrm{BMO}^c_\mathcal{R}}, \|f^*\|_{\mathrm{BMO}^c_\mathcal{R}} \big\}$$

with the column norm given by

$$\|f\|_{\mathrm{BMO}^c_\mathcal{R}} \ = \ \sup_{Q \in \mathcal{Q}} \Big\| \Big(-\!\!\!\!\!\!\int_Q (f(x) - f_Q)^* (f(x) - f_Q) \, dx \Big)^{\frac12} \Big\|_\mathcal{M}$$

where $\mathcal{Q}$ denotes the set of cubes in $\mathbb{R}^n$ with sides parallel to the axes.

Lemma 1.3. *We have* $T : L_\infty(\mathcal{R}_1) \to \mathrm{BMO}_{\mathcal{R}_2}^c$ *provided*

$$\text{i)}\ \Big\| \Big(\int_{\mathbb{R}^n} |Tf(x)|^2 \, dx \Big)^{\frac12} \Big\|_{\mathcal{M}_2} \lesssim \Big\| \Big(\int_{\mathbb{R}^n} |f(x)|^2 \, dx \Big)^{\frac12} \Big\|_{\mathcal{M}_1},$$

$$\text{ii)}\ \operatorname*{ess\,sup}_{x_1,x_2} \int_{|x_1-y|>2|x_1-x_2|} \big\| k(x_1,y) - k(x_2,y) \big\|_{\mathcal{B}(\mathcal{M}_1,\mathcal{M}_2)} \, dy < \infty.$$

Proof. We first observe that

$$\|g\|_{\mathrm{BMO}_\mathcal{R}^c} \sim_2 \sup_{Q \in \mathcal{Q}} \inf_{a_Q \in \mathcal{M}^\dagger} \Big\| \Big(\frac{1}{|Q|} \int_Q \big| g(x) - a_Q \big|^2 dx \Big)^{\frac12} \Big\|_{\mathcal{M}},$$

where $\mathcal{M}^\dagger$ stands for the algebra of operators affiliated with $\mathcal{M}$. Indeed,

$$\Big\| \Big(\fint_Q \big| g(x) - g_Q \big|^2 dx \Big)^{\frac12} \Big\|_{\mathcal{M}} \le \Big\| \Big(\fint_Q \big| g(x) - a_Q \big|^2 dx \Big)^{\frac12} \Big\|_{\mathcal{M}} + \big\| a_Q - g_Q \big\|_{\mathcal{M}}$$

and Kadison-Schwartz inequality for the conditional expectation $u(g) = g_Q \otimes \mathbf{1}_{\mathbb{R}^n}$ gives rise to $u(h)^* u(h) \le u(h^* h)$ for the function $h(x) = g(x) - a_Q$. Therefore, we obtain

$$\big\| a_Q - g_Q \big\|_{\mathcal{M}} = \big\| u(h)^* u(h) \big\|_{\mathcal{R}}^{\frac12} \le \big\| \sqrt{u(h^* h)} \big\|_{\mathcal{R}} = \Big\| \Big(\fint_Q \big| g(x) - a_Q \big|^2 dx \Big)^{\frac12} \Big\|_{\mathcal{M}}.$$

This proves the upper estimate, the lower estimate is clear. Now, given $f \in L_\infty(\mathcal{R}_1)$ and a ball Q, we set as usual $f_1 = f \chi_{5Q}$ and $f_2 = f - f_1$ where $5Q$ denotes the ball concentric to Q whose radius is 5 times the radius of Q. Then we pick $a_Q = \fint_Q Tf_2(x) \, dx$. It therefore suffices to prove

$$\mathrm{A} + \mathrm{B} = \Big\| \Big(\fint_Q \big| Tf_1(x) \big|^2 dx \Big)^{\frac12} \Big\|_{\mathcal{M}_2} + \Big\| \Big(\fint_Q \big| Tf_2(x) - a_Q \big|^2 dx \Big)^{\frac12} \Big\|_{\mathcal{M}_2} \lesssim \|f\|_{L_\infty(\mathcal{R}_1)}.$$

According to the L_2-column condition i), we find

$$\mathrm{A} \le \frac{1}{\sqrt{|Q|}} \Big\| \Big(\int_{5Q} \big| f(x) \big|^2 dx \Big)^{\frac12} \Big\|_{\mathcal{M}_1} \le 5^n \, \|f\|_{L_\infty(\mathcal{R}_1)}.$$

On the other hand, since $\mathrm{supp}_{\mathbb{R}^n} f_2 \cap Q = \emptyset$ we have for $x \in Q$

$$Tf_2(x) - a_Q = \fint_Q \big(Tf_2(x) - Tf_2(z) \big) \, dz = \fint_Q \int_{\mathbb{R}^n} \big(k(x,y) - k(z,y) \big) \big(f_2(y) \big) \, dy \, dz.$$

Using again the Kadison-Schwartz inequality, this gives rise to

$$\begin{aligned}
\mathrm{B} &= \Big\| \Big(\fint_Q \big| Tf_2(x) - a_Q \big|^2 dx \Big)^{\frac12} \Big\|_{\mathcal{M}_2} \\
&\le \Big(\fint_Q \fint_Q \Big\| \int_{\mathbb{R}^n} \big(k(x,y) - k(z,y) \big) \big(f_2(y) \big) \, dy \Big\|_{\mathcal{M}_2}^2 dz \, dx \Big)^{\frac12} \\
&\le \Big(\fint_Q \fint_Q \Big[\int_{\mathbb{R}^n \setminus 5Q} \big\| k(x,y) - k(z,y) \big\|_{\mathcal{B}(\mathcal{M}_1,\mathcal{M}_2)} \, dy \Big]^2 dz \, dx \Big)^{\frac12} \|f\|_{L_\infty(\mathcal{R}_1)} \\
&\le \Big(\operatorname*{ess\,sup}_{x,z \in \mathbb{R}^n} \int_{|x-y|>2|x-z|} \big\| k(x,y) - k(z,y) \big\|_{\mathcal{B}(\mathcal{M}_1,\mathcal{M}_2)} \, dy \Big) \|f\|_{L_\infty(\mathcal{R}_1)}. \qquad \square
\end{aligned}$$

Remark 1.4. The L_2-boundedness condition i) reduces to the classical one when $\mathcal{M}_1 = \mathcal{M}_2$ and $k(x,y)$ acts on $f(y)$ by left multiplication. Indeed, if we assume that T is bounded on $L_2(\mathcal{R})$ and use $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ for $\mathcal{H} = L_2(\mathcal{M})$

$$\begin{aligned}
&\Big\| \Big(\int_{\mathbb{R}^n} |Tf(y)|^2 \, dy \Big)^{\frac12} \Big\|_{\mathcal{M}} \\
&\quad = \sup_{\|h\| \le 1} \Big(\int_{\mathbb{R}^n} \big\langle h, |Tf(y)|^2 h \big\rangle_{\mathcal{H}} \, dy \Big)^{\frac12} \\
&\quad = \sup_{\|h\| \le 1} \big\| T(f\,(\mathbf{1}_{\mathbb{R}^n} \otimes h)) \big\|_{L_2(\mathcal{R})} \ \lesssim \ \sup_{\|h\| \le 1} \big\| f\,(\mathbf{1}_{\mathbb{R}^n} \otimes h) \big\|_{L_2(\mathcal{R})} \\
&\quad = \sup_{\|h\| \le 1} \Big(\int_{\mathbb{R}^n} \big\langle h, |f(y)|^2 h \big\rangle_{\mathcal{H}} \, dy \Big)^{\frac12} \ = \ \Big\| \Big(\int_{\mathbb{R}^n} |f(y)|^2 \, dy \Big)^{\frac12} \Big\|_{\mathcal{M}}.
\end{aligned}$$

This is false for other operator kernels and our L_2-condition seems the natural one.

Remark 1.5. Since $M_m(\mathrm{BMO}_{\mathcal{R}}) = \mathrm{BMO}_{M_m(\mathcal{R})}$, it suffices to replace $\mathcal{M}$ by $M_m(\mathcal{M})$ everywhere, amplify all the involved maps by tensorizing with id_{M_m} and require that the hypotheses hold with m-independent constants to deduce complete boundedness in the statement above.

1.4. **Nonequivariant CZO's.** Set $\Lambda^\dagger f = (\Lambda f^*)^*$ for any mapping Λ. In the nonequivariant setting, the arguments are not row/column symmetric because the map $(T \rtimes id_{\mathrm{G}})^\dagger$ is not similar to $T \rtimes id_{\mathrm{G}}$. This will be specially relevant in the L_2-boundedness conditions that we obtain. Indeed, we have

$$\begin{aligned}
(T \rtimes id_{\mathrm{G}})^\dagger \big(\sum\nolimits_g f_g \lambda(g) \big) \ &= \ \Big[(T \rtimes id_{\mathrm{G}}) \big(\sum\nolimits_g \alpha_{g^{-1}}(f_g^*) \lambda(g^{-1}) \big) \Big]^* \\
&= \sum\nolimits_g \alpha_g \big(T(\alpha_{g^{-1}}(f_g^*))^* \big) \lambda(g) = \sum\nolimits_g \alpha_g T^\dagger \alpha_{g^{-1}}(f_g) \lambda(g).
\end{aligned}$$

Thus, $(T \rtimes id_{\mathrm{G}})^\dagger$ is a map of the form $\sum_g f_g \lambda(g) \mapsto \sum_g T_g(f_g) \lambda(g)$ and recalling the embedding $j : \mathcal{M} \rtimes \mathrm{G} \to \mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathrm{G}))$, we see that

$$\begin{aligned}
j\big(\sum\nolimits_g T_g(f_g) \lambda(g) \big) \ &= \ \sum\nolimits_{g,h} \alpha_{g^{-1}}(T_{gh^{-1}}(f_{gh^{-1}})) \otimes e_{g,h} \\
&= \ \Big(\alpha_{g^{-1}} T_{gh^{-1}} \alpha_g \Big) \bullet j\big(\sum\nolimits_g f_g \lambda(g) \big) \ = \ \Phi\Big(j\big(\sum\nolimits_g f_g \lambda(g) \big) \Big),
\end{aligned}$$

where the $\bullet$ stands for the Schur product of matrices. In the result below, we will use the terminology of Lemma 1.3. Moreover, if $\mathcal{S} = (S_t)_{t \ge 0}$ denotes the heat semigroup on $\mathbb{R}^n$ we will write $\mathcal{S}_2 = (S_t \otimes id_{\mathcal{M}_2})_{t \ge 0}$ and

$$\widehat{\mathcal{M}}_j = \mathcal{M}_j \bar{\otimes} \mathcal{B}(\ell_2(\mathrm{G})).$$

Lemma 1.6. *Let* $\mathrm{G} \curvearrowright L_\infty(\mathbb{R}^n)$ *by an action* α *implemented by measure preserving transformations, so that* $\alpha_g f(x) = f(\beta_{g^{-1}} x)$*. Let us consider a family of* CZO's *formally given by* $T_g f(x) = \int_{\mathbb{R}^n} k_g(x,y)(f(y)) \, dy$ *for* $g \in \mathrm{G}$*. Then*

$$\sum\nolimits_g f_g \lambda(g) \mapsto \sum\nolimits_g T_g(f_g) \lambda(g)$$

is a cb-map $\mathcal{R}_1 \rtimes \mathrm{G} \to \mathrm{BMO}^c_{\mathcal{S}_{2\rtimes}}(\mathcal{R}_2 \rtimes \mathrm{G})$ *provided the following conditions hold*

i) *L_2-column condition,*

$$\Big\| \Big(\int_{\mathbb{R}^n} \big| (T_{gh^{-1}}) \bullet \rho \big|^2 (x) \, dx \Big)^{\frac{1}{2}} \Big\|_{\widehat{\mathcal{M}}_2} \ \lesssim_{cb} \ \Big\| \Big(\int_{\mathbb{R}^n} |\rho|^2 (x) \, dx \Big)^{\frac{1}{2}} \Big\|_{\widehat{\mathcal{M}}_1}.$$

ii) *Smoothness condition for the kernel,*

$$\operatorname*{ess\,sup}_{x_1,x_2} \int_{|x_1-y|>2|x_1-x_2|} \big\| K(x_1,y) - K(x_2,y) \big\|_{\mathcal{CB}(\widehat{\mathcal{M}}_1, \widehat{\mathcal{M}}_2)} \, dy \ < \ \infty,$$

where $K(x,y) = \sum_{g,h} k_{gh^{-1}}(\beta_g x, \beta_g y) \otimes e_{g,h}$ acts as a Schur multiplier.

Proof. According to Lemma 1.1 and since

$$j \Big(\sum\nolimits_g T_g(f_g) \lambda(g) \Big) \ = \ \Phi \Big(j \big(\sum\nolimits_g f_g \lambda(g) \big) \Big),$$

it suffices to show that $\Phi : \widehat{\mathcal{R}}_1 \to \mathrm{BMO}_{\mathcal{S}_\otimes}^c(\widehat{\mathcal{R}}_2)$ is a cb-map where $\widehat{\mathcal{R}}_j = \mathcal{R}_j \bar{\otimes} \mathcal{B}(\ell_2(\mathrm{G}))$ and $\mathcal{S}_\otimes = (S_t \otimes id_{\widehat{\mathcal{M}}_2})_{t \ge 0}$. Letting $\rho = \sum_{g,h} a_{g,h} \otimes e_{g,h}$ with $a_{g,h} \in \mathcal{R}_1$, we find that

$$\begin{aligned} \Phi(\rho)(x) &= \sum\nolimits_{g,h} \alpha_{g^{-1}} \int_{\mathbb{R}^n} k_{gh^{-1}}(x,y) (a_{g,h}(\beta_{g^{-1}}(y))) \, dy \otimes e_{g,h} \\ &= \sum\nolimits_{g,h} \int_{\mathbb{R}^n} k_{gh^{-1}}(\beta_g x, \beta_g y)(a_{g,h}(y)) \, dy \otimes e_{g,h} = \int_{\mathbb{R}^n} K(x,y)(\rho(y)) \, dy. \end{aligned}$$

Therefore, we may regard Φ as a semicommutative CZO and apply Lemma 1.3 together with Remark 1.5. First, we note that the L_2-boundedness assumption means that the map $\Phi : L_\infty(\widehat{\mathcal{M}}_1; L_2^c(\mathbb{R}^n)) \to L_\infty(\widehat{\mathcal{M}}_2; L_2^c(\mathbb{R}^n))$ is cb. However, we have

$$\Phi(\rho) = (\alpha_{g^{-1}} T_{gh^{-1}} \alpha_g) \bullet \rho = (\alpha_{g^{-1}}) \bullet (T_{gh^{-1}}) \bullet (\alpha_g) \bullet \rho.$$

Using that β is measure preserving, we find that the Schur product map

$$\sum\nolimits_{g,h} a_{g,h} \otimes e_{g,h} \mapsto \sum\nolimits_{g,h} \alpha_g(a_{g,h}) \otimes e_{g,h}$$

is a cb-isometry on $L_\infty(\widehat{\mathcal{M}}_1; L_2^c(\mathbb{R}^n))$, and the same holds taking $(\alpha_{g^{-1}}, \mathcal{M}_2)$ in place of $(\alpha_g, \mathcal{M}_1)$. This shows that the L_2-boundedness condition given in Lemma 1.3 for Φ reduces to the cb-boundedness condition in the statement. On the other hand, the smoothness condition matches exactly that of Lemma 1.3. $\square$

Lemma 1.7. *Let $T : \mathcal{R} \to \mathrm{BMO}_{\mathcal{S}_\otimes}(\mathcal{R})$ given by*

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y) f(y) \, dy \quad (x \notin \mathrm{supp}_{\mathbb{R}^n} f)$$

with $\mathcal{S}_\otimes = (S_t \otimes id_\mathcal{M})_{t \ge 0}$, $f \in \mathcal{R} = L_\infty(\mathbb{R}^n) \bar{\otimes} \mathcal{M}$ and $k(x,y) \in \mathcal{M}$ acting on f by left multiplication. Then, the cross product extension $T \rtimes id_\mathrm{G} : \mathcal{R} \rtimes \mathrm{G} \to \mathrm{BMO}_{\mathcal{S}_\rtimes}^c(\mathcal{R} \rtimes \mathrm{G})$ is cb-bounded provided

i) *$T : L_2(\mathcal{R}) \to L_2(\mathcal{R})$ bounded,*

ii) $$\operatorname*{ess\,sup}_{x_1,x_2} \int_{|x_1-y|>2|x_1-x_2|} \sup_{g \in \mathrm{G}} \big\| k(\beta_g x_1, \beta_g y) - k(\beta_g x_2, \beta_g y) \big\|_\mathcal{M} \, dy \ < \ \infty.$$

Proof. This is a particular case of Lemma 1.6 with $T_g = T$ for all g, $\mathcal{M}_1 = \mathcal{M}_2$ and the kernel acting by left multiplication. The L_2-column condition clearly reduces to

$$\Big\| \Big(\int_{\mathbb{R}^n} \big|Tf(x)\big|^2 \, dx \Big)^{\frac{1}{2}} \Big\|_{\mathcal{M}} \lesssim_{cb} \Big\| \Big(\int_{\mathbb{R}^n} |f(x)|^2 \, dx \Big)^{\frac{1}{2}} \Big\|_{\mathcal{M}},$$

which in turn reduces to $L_2(\mathcal{R})$-boundedness by means of Remark 1.4 and the fact that boundedness is equivalent to cb-boundedness on $L_2 = \mathrm{OH}$, see e.g. [64]. On the other hand, $K(x,y) = \sum_{g,h} k(\beta_g x, \beta_g y) \otimes e_{g,h}$ since $T_g = T$. Hence, we deduce that

$$\begin{aligned} K(x,y)(f(y)) &= \Big(k(\beta_g x, \beta_g y) \Big) \bullet \Big(f_{g,h}(y) \Big) \\ &= \Big[\sum\nolimits_g k(\beta_g x, \beta_g y) \otimes e_{gg} \Big] \Big[\sum\nolimits_{g,h} f_{g,h}(y) \otimes e_{g,h} \Big]. \end{aligned}$$

In particular, regarding $K(x,y)$ as a left multiplication map (not a Schur multiplier) it is a diagonal matrix in $\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathrm{G}))$ with entries $k(\beta_g x, \beta_g y)$. Therefore, we may easily rewrite the Hörmander smoothness condition for the kernel in Lemma 1.6 as it is written in the statement. This completes the proof. $\square$

2. Hörmander-Mihlin multipliers

We now study Fourier multipliers over the group von Neumann algebra of an arbitrary discrete group G. In the language of quantum groups, these algebras are regarded as the compact dual of G. Our main result is a cocycle form of Hörmander-Mihlin multiplier theorem in this setting.

2.1. **Length functions and cocycles.** A *left cocycle* associated to a discrete group G is a triple $(\mathcal{H}, \alpha, b)$ formed by a Hilbert space $\mathcal{H}$, an isometric action $\alpha : \mathrm{G} \to \mathrm{Aut}(\mathcal{H})$ and a map $b : \mathrm{G} \to \mathcal{H}$ so that $\alpha_g(b(h)) = b(gh) - b(g)$. A *right cocycle* satisfies the relation $\alpha_g(b(h)) = b(hg^{-1}) - b(g^{-1})$ instead. In this paper a (cocycle) *length function* $\psi : \mathrm{G} \to \mathbb{R}_+$ is any symmetric conditionally negative function vanishing at the identity of G, as defined in the Introduction. The fact that any length function takes values in $\mathbb{R}_+$ is easily justified. Any cocycle $(\mathcal{H}, \alpha, b)$ can be identified with an affine representation

$$g \in \mathrm{G} \mapsto \left(\begin{array}{cc} \alpha_g & b(g) \\ 0 & 1 \end{array} \right) \in \mathrm{Aff}(\mathcal{H}).$$

In what follows, we only consider cocycles with values in real Hilbert spaces. Note that $\mathrm{Aut}(\mathcal{H})$ is the orthogonal group on $\mathcal{H}$ and $\mathrm{Aff}(\mathcal{H}) \simeq \mathcal{H} \rtimes \mathrm{Aut}(\mathcal{H})$. Any cocycle $(\mathcal{H}, \alpha, b)$ gives rise to an associated length function $\psi_b(g) = \langle b(g), b(g) \rangle_{\mathcal{H}}$, as it can be checked by the reader. Reciprocally, any length function ψ gives rise to a left and a right cocycle. This is a standard application of the ideas around Schoenberg's theorem [72], which claims that $\psi : \mathrm{G} \to \mathbb{R}_+$ is a length function if and only if the mappings $S_{\psi,t}(\lambda(g)) = \exp(-t\psi(g))\lambda(g)$ extend to a semigroup of unital completely positive maps on $\mathcal{L}(\mathrm{G})$. Let us collect these well-known results.

Lemma 2.1. *If $\psi : \mathrm{G} \to \mathbb{R}_+$ is a length function:*

i) *The forms*

$$K_\psi^1(g,h) = \frac{\psi(g) + \psi(h) - \psi(g^{-1}h)}{2},$$

$$K_\psi^2(g,h) \ = \ \frac{\psi(g)+\psi(h)-\psi(gh^{-1})}{2},$$

define positive matrices on $\mathrm{G}\times\mathrm{G}$ *and lead to*

$$\Big\langle \sum\nolimits_g a_g\delta_g, \sum\nolimits_h b_h\delta_h \Big\rangle_{\psi,j} = \sum\nolimits_{g,h} a_g K_\psi^j(g,h) b_h$$

on the group algebra $\mathbb{R}[\mathrm{G}]$ *of finitely supported real functions on* G.

ii) *Let* $\mathcal{H}_\psi^j$ *be the Hilbert space completion of*

$$\big(\mathbb{R}[\mathrm{G}]/N_\psi^j, \langle\cdot,\cdot\rangle_{\psi,j}\big) \quad \textit{with} \quad N_\psi^j = \textit{null space of } \langle\cdot,\cdot\rangle_{\psi,j}.$$

If we consider the mapping $b_\psi^j : g\in\mathrm{G} \mapsto \delta_g + N_\psi^j \in \mathcal{H}_\psi^j$

$$\begin{aligned} \alpha_{\psi,g}^1\Big(\sum_{h\in\mathrm{G}} a_h b_\psi^1(h)\Big) &= \sum_{h\in\mathrm{G}} a_h\big(b_\psi^1(gh)-b_\psi^1(g)\big), \\ \alpha_{\psi,g}^2\Big(\sum_{h\in\mathrm{G}} a_h b_\psi^2(h)\Big) &= \sum_{h\in\mathrm{G}} a_h\big(b_\psi^2(hg^{-1})-b_\psi^2(g^{-1})\big), \end{aligned}$$

determine isometric actions $\alpha_\psi^j : \mathrm{G}\to\mathrm{Aut}(\mathcal{H}_\psi^j)$ *of* G *on* $\mathcal{H}_\psi^j$.

iii) *Imposing the discrete topology on* $\mathcal{H}_\psi^j$, *the semidirect product* $\mathrm{G}_\psi^j = \mathcal{H}_\psi^j \rtimes \mathrm{G}$ *becomes a discrete group and we find the following group homomorphisms*

$$\begin{aligned} \pi_\psi^1 : g\in\mathrm{G} &\mapsto b_\psi^1(g)\rtimes g\in\mathrm{G}_\psi^1, \\ \pi_\psi^2 : g\in\mathrm{G} &\mapsto b_\psi^2(g^{-1})\rtimes g\in\mathrm{G}_\psi^2. \end{aligned}$$

The previous lemma allows us to introduce two pseudo-metrics on our discrete group G in terms of the length function ψ. Indeed, a short calculation leads to the crucial identities

$$\psi(g^{-1}h) \ = \ \big\langle b_\psi^1(g)-b_\psi^1(h), b_\psi^1(g)-b_\psi^1(h)\big\rangle_{\psi,1} \ = \ \big\|b_\psi^1(g)-b_\psi^1(h)\big\|_{\mathcal{H}_\psi^1}^2,$$

$$\psi(gh^{-1}) \ = \ \big\langle b_\psi^2(g)-b_\psi^2(h), b_\psi^2(g)-b_\psi^2(h)\big\rangle_{\psi,2} \ = \ \big\|b_\psi^2(g)-b_\psi^2(h)\big\|_{\mathcal{H}_\psi^2}^2.$$

In particular,

$$\mathrm{dist}_1(g,h) = \sqrt{\psi(g^{-1}h)} = \|b_\psi^1(g)-b_\psi^1(h)\|_{\mathcal{H}_\psi^1}$$

defines a pseudo-metric on G, which becomes a metric when the cocycle map is injective. Similarly, we may work with $\mathrm{dist}_2(g,h) = \sqrt{\psi(gh^{-1})}$. The following elementary observation will be crucial for what follows.

Lemma 2.2. *Let* $(\mathcal{H}_1,\alpha_1,b_1)$ *and* $(\mathcal{H}_2,\alpha_2,b_2)$ *be a left and a right cocycle on* G. *Assume that the associated length functions* ψ_{b_1} *and* ψ_{b_2} *coincide, then we find an isometric isomorphism*

$$\Lambda_{12} : b_1(g)\in\mathcal{H}_1 \mapsto b_2(g^{-1})\in\mathcal{H}_2.$$

In particular, given a length function ψ *we see that* $\mathcal{H}_\psi^1 \simeq \mathcal{H}_\psi^2$ *via* $b_\psi^1(g)\mapsto b_\psi^2(g^{-1})$.

Proof. By polarization, we see that

$$\big\langle b_1(g), b_1(h)\big\rangle_{\mathcal{H}_1} = \frac12\Big(\|b_1(g)\|_{\mathcal{H}_1}^2 + \|b_1(h)\|_{\mathcal{H}_1}^2 - \|b_1(g)-b_1(h)\|_{\mathcal{H}_1}^2\Big).$$

Since $b_1(g) - b_1(h) = \alpha_{1,h}(b_1(h^{-1}g))$, we obtain

$$\begin{aligned}\big\langle b_1(g), b_1(h) \big\rangle_{\mathcal{H}_1} &= \frac{\psi_{b_1}(g) + \psi_{b_1}(h) - \psi_{b_1}(g^{-1}h)}{2} \\ &= \frac{\psi_{b_2}(g) + \psi_{b_2}(h) - \psi_{b_2}(g^{-1}h)}{2} \ = \ \big\langle b_2(g^{-1}), b_2(h^{-1}) \big\rangle_{\mathcal{H}_2}.\end{aligned}$$

The last identity uses polarization and $b_2(g^{-1}) - b_2(h^{-1}) = \alpha_{2,h}(b_2(g^{-1}h))$. $\square$

2.2. **Smooth Fourier multipliers.** We are now ready to prove our extension of Hörmander/Mihlin's sufficient condition for Fourier multipliers to arbitrary discrete groups. The ideas leading to the next result probably go back to Hörmander, but we could not find the specific statement given below in the literature. We provide a proof based on Stein's approach to these questions in his book [74].

Lemma 2.3. *Let $k_{\widetilde{m}}$ be a tempered distribution on $\mathbb{R}^n$ which coincides with a locally integrable function on $\mathbb{R}^n \setminus \{0\}$. Let $\widetilde{m}$ stand for its Fourier transform $\widetilde{m} = \widehat{k_{\widetilde{m}}}$. Then we obtain the following results*:

i) *If $|\partial_\xi^\beta \widetilde{m}(\xi)| \le c_n |\xi|^{-|\beta|}$ for all $|\beta| \le n+2$*

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \int_{|y|>2|x|} \sup_{g \in \mathrm{G}} \big| k_{\widetilde{m}}(\beta_g y - \beta_g x) - k_{\widetilde{m}}(\beta_g y) \big| \, dy \ < \ \infty.$$

ii) *If $|\partial_\xi^\beta \widetilde{m}(\xi)| \le c_n |\xi|^{-|\beta|}$ for all $|\beta| \le [\frac{n}{2}] + 1$, the operator*

$$T_{k_{\widetilde{m}}}\Big(\sum_{g,h \in \mathrm{G}} f_{gh} \otimes e_{gh} \Big) \ = \ \sum_{g,h \in \mathrm{G}} \int k_{\widetilde{m}}(\beta_g x - \beta_g y) f_{gh}(y) \, dy \otimes e_{gh}$$

extends to a cb-map from $L_\infty(\mathbb{R}^n; \mathcal{B}(\ell_2(\mathrm{G})))$ to $\mathrm{BMO}_c(\mathbb{R}^n; \mathcal{B}(\ell_2(\mathrm{G})))$.

Proof. For i), it suffices to show that $|\nabla k_{\widetilde{m}}(z)| \lesssim |z|^{-(n+1)}$. Let $\eta \in \mathcal{C}^\infty(\mathbb{R}^n)$ with $\chi_{\mathrm{B}_1(0)} \le \eta \le \chi_{\mathrm{B}_2(0)}$ and take $\delta(\xi) = \eta(\xi) - \eta(2\xi)$ so that $\sum_{j \in \mathbb{Z}} \delta(2^{-j}\xi) = 1$ for all $\xi \neq 0$. This gives rise to $\widetilde{m}(\xi) = \sum_j \widetilde{m}(\xi)\delta(2^{-j}\xi) = \sum_j \widetilde{m}_j(\xi)$ and we set

$$k_{\widetilde{m}}^j(x) = \int_{\mathbb{R}^n} \widetilde{m}_j(\xi) e^{2\pi i \langle x, \xi \rangle} \, d\xi.$$

We have $\sum_j k_{\widetilde{m}}^j \to k_{\widetilde{m}}$ as distributions, so that it suffices to estimate $\sum_j |\partial_x^\alpha k_{\widetilde{m}}^j(x)|$ for any $x \neq 0$ and any multi-index α with $|\alpha| = 1$. Now we claim that a) $\Rightarrow$ b) with

a) $\big|\partial_\xi^\beta \widetilde{m}(\xi)\big| \le c_M |\xi|^{-|\beta|}$ for all multi-index β s.t. $0 \le |\beta| \le M$.

b) $\big|\partial_x^\alpha k_{\widetilde{m}}^j(x)\big| \le c_M |x|^{-M} 2^{j(n-M+1)}$ for all multi-index α s.t. $|\alpha| = 1$.

Let us first see how the assertion follows from the claim. Indeed, we know from our hypotheses that a) holds for any $0 \le M \le n+2$. If we apply our claim for $M = 0$ on those j's for which $2^j \le |x|^{-1}$ and we apply it for $M = n+2$ on those j's for which $2^j > |x|^{-1}$, we find

$$\sum_{j \in \mathbb{Z}} \big|\partial_x^\alpha k_{\widetilde{m}}^j(x)\big| \ \lesssim \ \sum_{2^j \le |x|^{-1}} 2^{j(n+1)} + \frac{1}{|x|^{n+2}} \sum_{2^j > |x|^{-1}} 2^{-j} \ \sim \ \frac{1}{|x|^{n+1}}.$$

To prove our claim, we use the properties of the Fourier transform to get

$$(-2\pi i x)^\gamma \partial_x^\alpha k_{\widetilde{m}}^j(x) \ = \ \int_{\mathbb{R}^n} \partial_\xi^\gamma \big[(2\pi i \xi)^\alpha \widetilde{m}_j(\xi) \big] e^{2\pi i \langle x, \xi \rangle} \, d\xi.$$

On the other hand, using condition a) it is not difficult to check that we have

$$\Big|\partial_\xi^\gamma\big[(2\pi i\xi)^\alpha\widetilde{m}_j(\xi)\big]\Big| \le \sum_{\gamma_1+\gamma_2=\gamma} c_{\gamma_1\gamma_2}\Big|\partial_\xi^{\gamma_1}\big((2\pi i\xi)^\alpha\big)\partial_\xi^{\gamma_2}\widetilde{m}_j(\xi)\Big| \lesssim |\xi|^{1-|\gamma|}.$$

Moreover, since $\widetilde{m}_j$ is supported by an annulus of radius $\sim 2^j$, we conclude that

$$\Big|\int_{\mathbb{R}^n}\partial_\xi^\gamma\big[(2\pi i\xi)^\alpha\widetilde{m}_j(\xi)\big]e^{2\pi i\langle x,\xi\rangle}\,d\xi\Big| \lesssim 2^{jn}2^{j(1-|\gamma|)}.$$

Given $x\in\mathbb{R}^n$ there exists a multi-index γ such that $|\gamma|=M$ and $|x^\gamma|\sim|x|^M$. Hence, taking such a multi-index γ in the identity above we deduce our claim. Let us now prove ii). If $f=\sum_g f_{gh}\otimes e_{gh}$, we have

$$\begin{aligned}
&\Big\|\frac{1}{|Q|}\int_Q\big|T_{k_{\widetilde{m}}}(f)-(T_{k_{\widetilde{m}}}(f))_Q\big|^2\Big\|_{\mathcal{B}(\ell_2(\mathrm{G}))}^{\frac12}\\
&\qquad = \sup_{\|\xi\|_{\ell_2(\mathrm{G})}=1}\Big(\frac{1}{|Q|}\int_Q\big\|T_{k_{\widetilde{m}}}(f)\xi-(T_{k_{\widetilde{m}}}(f)\xi)_Q\big\|_{\ell_2(\mathrm{G})}^2\Big)^{\frac12}\\
&\qquad = \sup_{\|\xi\|_{\ell_2(\mathrm{G})}=1}\Big(\frac{1}{|Q|}\int_Q\big\|T_{k_{\widetilde{m}}}(f\xi)-(T_{k_{\widetilde{m}}}(f\xi))_Q\big\|_{\ell_2(\mathrm{G})}^2\Big)^{\frac12},
\end{aligned}$$

where $f\xi=\sum_g(\sum_h f_{gh}\xi_h)\otimes e_g$ satisfies

$$\|f\xi\|_{L_\infty(\mathbb{R}^n;\ell_2(\mathrm{G}))}\le\|f\|_\infty.$$

The problem is then reduced to show that the restriction of $T_{\widetilde{m}}$ to column matrices extends to a bounded map $L_\infty(\mathbb{R}^n;\ell_2(\mathrm{G}))\to\mathrm{BMO}(\mathbb{R}^n;\ell_2(\mathrm{G}))$. Let us decompose $f\xi$ in the usual way

$$f_{\xi,1}=f\xi\chi_{5Q}\quad\text{and}\quad f_{\xi,2}=f\xi-f_{\xi,1}.$$

By the $L_2(\mathbb{R}^n;\ell_2(\mathrm{G}))$ boundedness of $T_{\widetilde{m}}$, we have

$$\begin{aligned}
&\Big\|\frac{1}{|Q|}\int_Q\big|T_{k_{\widetilde{m}}}(f)-(T_{k_{\widetilde{m}}}(f))_Q\big|^2\Big\|_{\mathcal{B}(\ell_2(\mathrm{G}))}^{\frac12}\\
&\qquad \lesssim \sup_{\|\xi\|_{\ell_2(\mathrm{G})}=1}\Big(\frac{1}{|Q|}\int_Q\|T_{k_{\widetilde{m}}}(f_{\xi,1})\|_{\ell_2(\mathrm{G})}^2\Big)^{\frac12}\\
&\qquad + \sup_{\|\xi\|_{\ell_2(\mathrm{G})}=1}\Big(\frac{1}{|Q|}\int_Q\big\|T_{k_{\widetilde{m}}}(f_{\xi,2})-(T_{k_{\widetilde{m}}}(f_{\xi,2}))_Q\big\|_{\ell_2(\mathrm{G})}^2\Big)^{\frac12}\\
&\qquad \lesssim \|f\|_\infty+\sup_{\substack{\|\xi\|_{\ell_2(\mathrm{G})}=1\\ x,z\in Q}}\big\|T_{k_{\widetilde{m}}}(f_{\xi,2})(x)-T_{k_{\widetilde{m}}}(f_{\xi,2})(z)\big\|_{\ell_2(\mathrm{G})}.
\end{aligned}$$

The last term on the right hand side can be rewritten as follows

$$\begin{aligned}
&\sup_{\substack{\|\xi\|_2,\|\eta\|_2=1\\ x,z\in Q}}\Big|\int_{|x-y|\ge2|x-z|}\Big\langle\big(\eta_g[k_{\widetilde{m}}(\beta_gx-\beta_gy)-k_{\widetilde{m}}(\beta_gz-\beta_gy)]\big),f_{\xi,2}(y)\Big\rangle\,dy\Big|\\
&\le \sup_{\substack{\|\eta\|_2=1\\ x,z\in Q}}\int_{|x-y|\ge2|x-z|}\Big\|\big(\eta_g[k_{\widetilde{m}}(\beta_gx-\beta_gy)-k_{\widetilde{m}}(\beta_gz-\beta_gy)]\big)\Big\|_{\ell_2(\mathrm{G})}dy\ \|f\|_\infty.
\end{aligned}$$

Following a classical argument, it is easy to check that the last term in the inequality above is finite. In fact, arguing as for inequality (32) of [74, VI.4.4.2] —see also our estimates for i)— we may decompose $k_{\widetilde{m}} = \sum_j k_{\widetilde{m}}^j$ and conclude that

$$\int_{\mathbb{R}^n} |x|^{2M} \Big\| \sum_g \eta_g k_{\widetilde{m}}^j(\beta_g x) \otimes e_g \Big\|_{\ell_2(\mathrm{G})}^2 \, dx \, \lesssim \, 2^{j(n-2M)},$$

for any $0 \le M \le [\frac{n}{2}] + 1$ and any $j \in \mathbb{Z}$. The remaining part of the estimation is the same to that of [74, VI.4.4.2]. Thus, taking the supremum over Q we deduce the estimate for the norm of $T_{k_{\widetilde{m}}}$. The cb-norm is estimated similarly. □

Theorem 2.4. *Given a discrete group* G*, let*

$$T_m : \sum_{g \in \mathrm{G}} \widehat{f}(g) \lambda(g) \mapsto \sum_{g \in \mathrm{G}} m_g \widehat{f}(g) \lambda(g).$$

Let $\psi : \mathrm{G} \to \mathbb{R}_+$ be any length function and set $(\mathcal{H}_j, \alpha_j, b_j)$ for the left and right cocycles associated to it $(j = 1, 2)$. Assume $\dim \mathcal{H}_j = n < \infty$ and let $\widetilde{m}_j : \mathcal{H}_j \to \mathbb{C}$ be lifting multipliers for m, so that $m = \widetilde{m}_j \circ b_j$. Then, if $\widetilde{m}_j \in \mathcal{C}^{[\frac{n}{2}]+1}(\mathcal{H}_j \setminus \{0\})$ and

$$\big| \partial_\xi^\beta \widetilde{m}_j(\xi) \big| \, \le \, c_n |\xi|^{-|\beta|} \quad \textit{for all multi-index } \beta \textit{ s.t.} \quad |\beta| \le [\tfrac{n}{2}] + 1,$$

we find that $T_m : L_p(\widehat{\mathbb{G}}) \to L_p(\widehat{\mathbb{G}})$ for all $1 < p < \infty$ and $T_m : \mathcal{L}(\mathrm{G}) \to \mathrm{BMO}_{\mathcal{S}_\psi}$.

Proof. We divide it in several steps:

A. *Reduction to $L_\infty \to$ BMO.* Assume that the hypotheses imply $L_\infty \to$ BMO boundedness. Since the condition for $\beta = 0$ implies that $\widetilde{m}_1$ is bounded, the same holds for $m = \widetilde{m}_1 \circ b_1$ and we deduce the L_2 boundedness for T_m. The L_p boundedness for $2 < p < \infty$ follows by interpolation from [26]. Indeed, if we let J_p the projection map onto the complemented subspace

$$L_p^\circ(\widehat{\mathbb{G}}) = \Big\{ f \in L_p(\widehat{\mathbb{G}}) \, \big| \, \lim_{t \to \infty} S_{\psi,t} f = 0 \Big\},$$

we get from [26] that $J_p T_m : L_p(\widehat{\mathbb{G}}) \to L_p^\circ(\widehat{\mathbb{G}})$. However, $E_p = id_{L_p(\widehat{\mathbb{G}})} - J_p$ is the projection onto the fixed point subspace, which in this case is the closure of the span of $\lambda(g)$'s such that $\psi(g) = 0$. Since it is clear that $\mathrm{G}_0 = \{g \in \mathrm{G} \, | \, \psi(g) = 0\}$ is a subgroup of G, we deduce that E_p is a conditional expectation. This implies that $T_m = m_e E_p + J_p T_m$ is also bounded. To prove the case $1 < p < 2$, we proceed by duality since $T_m^* = T_{\overline{m}}$ and the argument above also applies to $\overline{m}$.

B. *Reduction to the column* BMO *estimate.* Assume now that $T_m : \mathcal{L}(\mathrm{G}) \to \mathrm{BMO}_{\mathcal{S}_\psi}^c$ holds under the hypotheses we have imposed in the statement. Then T_m is also a bounded map $L_\infty \to$ BMO. Indeed, the row BMO boundedness of T_m is equivalent to the column BMO boundedness of

$$T_m^\dagger \Big(\sum_{g \in \mathrm{G}} \widehat{f}(g) \lambda(g) \Big) = T_m \Big(\sum_{g \in \mathrm{G}} \overline{\widehat{f}(g)} \lambda(g^{-1}) \Big)^* = \sum_{g \in \mathrm{G}} \overline{m}_{g^{-1}} \widehat{f}(g) \lambda(g).$$

This shows that $T_m^\dagger = T_k$ with $k_g = \overline{m}_{g^{-1}} = \overline{\widetilde{m}}_j \circ b_j(g^{-1})$. According to Lemma 2.2, we deduce that $k_g = \widetilde{k}_j \circ b_j$ where $\widetilde{k}_1 = \overline{\widetilde{m}}_2 \circ \Lambda_{12}$ and $\widetilde{k}_2 = \overline{\widetilde{m}}_1 \circ \Lambda_{12}^{-1}$. Since Λ_{12} is an orthogonal transformation on $\mathbb{R}^n$ and the complex conjugation is harmless, it turns out that the $\widetilde{k}_j$'s satisfy one more time the same conditions as the $\widetilde{m}_j$'s and the assertion will follow if we can prove that the column BMO estimate holds.

C. *The key intertwining identities.* We will only work here with the left cocycle $(\mathcal{H}_1, \alpha_1, b_1)$. Let us write $\mathbb{R}^n_{\mathrm{disc}}$ for the discrete additive group on $\mathbb{R}^n$. According to the discrete topology imposed in Lemma 2.1 and since $\dim \mathcal{H}_1 = n$, this is a suitable realization of $\mathcal{H}_1$. The dual group is the Bohr compactification of $\mathbb{R}^n$ and $\mathcal{L}(\mathcal{H}_\psi)$ is the corresponding L_∞ space on it. Let λ_1 and $\lambda_\rtimes$ denote the left regular representations on $\mathcal{H}_1$ and $\mathrm{G}_\rtimes = \mathcal{H}_1 \rtimes \mathrm{G}$ respectively, while $\exp b_1(g)$ will stand for $\lambda_1(b_1(g)) \simeq \exp(2\pi i \langle b_1(g), \cdot \rangle)$. Consider the trace preserving, normal homomorphism given by $\pi_1 : \lambda(g) \in \mathcal{L}(\mathrm{G}) \mapsto \lambda_\rtimes(b_1(g) \rtimes g) \in \mathcal{L}(\mathrm{G}_\rtimes)$. It is very tempting and in fact very useful to use that $\mathcal{L}(\mathcal{H}_1)$ is commutative, by switching between the language of von Neumann algebras of discrete groups and semidirect products of von Neumann algebras. Indeed, it is a simple exercise to show that $\mathcal{L}(\mathrm{G}_\rtimes) \simeq \mathcal{L}(\mathcal{H}_1) \rtimes \mathrm{G}$. In particular, the embedding π_1 takes the following form

$$\pi_1 : \lambda(g) \in \mathcal{L}(\mathrm{G}) \mapsto \exp b_1(g) \lambda(g) \in \mathcal{L}(\mathcal{H}_1) \rtimes \mathrm{G}.$$

Let $\mathcal{S}_\rtimes = (S_{\rtimes,t})_{t \ge 0}$ denote the crossed product extension $S_{\rtimes,t} = S_t \rtimes id_{\mathrm{G}}$ of the heat semigroup on $\mathcal{H}_1 \simeq \mathbb{R}^n_{\mathrm{disc}}$. It is evident that the heat semigroup is G-equivariant with respect to any isometric action on $\mathcal{H}_1$. We now claim that the hypotheses on $\widetilde{m}_1$ imply that $T_\rtimes : \mathcal{L}(\mathcal{H}_1) \rtimes \mathrm{G} \to \mathrm{BMO}^c_{\mathcal{S}_\rtimes}$ is bounded, where

$$T_\rtimes \Big(\sum_{g \in \mathrm{G}} f_g \lambda(g) \Big) = \sum_{g \in \mathrm{G}} T_{\widetilde{m}_1}(f_g) \lambda(g) \quad \text{with} \quad T_{\widetilde{m}_1}(\exp b_1(h)) = \widetilde{m}_1(b_1(h)) \exp b_1(h).$$

Let us see how the assertion (which has been reduced to check the column-$\mathrm{BMO}_{\mathcal{S}_\psi}$ boundedness) follows from our claim. The key points are the intertwining identities

$$\pi_1 \circ S_{\psi,t} \ = \ S_{\rtimes,t} \circ \pi_1 \quad \text{and} \quad \pi_1 \circ T_m \ = \ T_\rtimes \circ \pi_1.$$

Indeed, it is easily checked that the first one follows from $\psi(g) = \langle b_1(g), b_1(g) \rangle_{\mathcal{H}_1}$ while the second one from $m_g = \widetilde{m}_1(b_1(g))$. We leave the reader to check it. We can now prove the assertion

$$\begin{aligned} \|T_m f\|_{\mathrm{BMO}^c_{\mathcal{S}_\psi}} & = \sup_{t>0} \Big\| S_{\psi,t} |T_m f|^2 - |S_{\psi,t} T_m f|^2 \Big\|_{\mathcal{L}(\mathrm{G})}^{\frac12} \\ & = \sup_{t>0} \Big\| \pi_1 \Big(S_{\psi,t} |T_m f|^2 - |S_{\psi,t} T_m f|^2 \Big) \Big\|_{\mathcal{L}(\mathcal{H}_1) \rtimes \mathrm{G}}^{\frac12} \\ & = \sup_{t>0} \Big\| S_{\rtimes,t} |T_\rtimes \pi_1 f|^2 - |S_{\rtimes,t} T_\rtimes \pi_1 f|^2 \Big\|_{\mathcal{L}(\mathcal{H}_1) \rtimes \mathrm{G}}^{\frac12} \\ & = \big\| T_\rtimes(\pi_1 f) \big\|_{\mathrm{BMO}^c_{\mathcal{S}_\rtimes}} \ \le \ c \, \big\| \pi_1 f \big\|_{\mathcal{L}(\mathcal{H}_1) \rtimes \mathrm{G}} \ = \ c \, \|f\|_{\mathcal{L}(\mathrm{G})}. \end{aligned}$$

D. BMO *extension of de Leeuw's theorem.* We will show that

i) $T_\rtimes : L_\infty(\mathbb{R}^n) \rtimes \mathrm{G} \to \mathrm{BMO}^c_{\mathcal{S}_\rtimes}(L_\infty(\mathbb{R}^n) \rtimes \mathrm{G})$,
ii) $T_\rtimes : L_\infty(\widehat{\mathbb{R}}^n_{\mathrm{disc}}) \rtimes \mathrm{G} \to \mathrm{BMO}^c_{\mathcal{S}_\rtimes}(L_\infty(\widehat{\mathbb{R}}^n_{\mathrm{disc}}) \rtimes \mathrm{G})$,

are equivalent when the crossed product $L_\infty(\mathbb{R}^n) \rtimes \mathrm{G}$ is also constructed from the action α_1. This reduces the proof of our claim to the Euclidean setting. Letting $\mathcal{S}$ denote the heat semigroup on $\mathbb{R}^n$, the proof will be more transparent by showing $T_{\widetilde{m}_1} : L_\infty(\mathbb{R}^n) \to \mathrm{BMO}_{\mathcal{S}} \Leftrightarrow T_{\widetilde{m}_1} : L_\infty(\widehat{\mathbb{R}}^n_{\mathrm{disc}}) \to \mathrm{BMO}_{\mathcal{S}}$. After that, we will point out the slight modifications needed to make the argument work after taking crossed products with G. The L_p analogue of this equivalence is a classical result

of de Leeuw [9]. Since $\mathbb{R}^n$ and $\mathbb{R}^n_{\mathrm{disc}}$ coincide as sets, we let Γ denote it. According to Pontryagin duality, the continuous characters on $\mathbb{R}^n$ and its compactification are both indexed by Γ. Write χ_γ and χ'_γ for the continuous characters on $\mathbb{R}^n$ and its Bohr compactification respectively. According to the construction of the Bohr compactification, we find a universal inclusion map $\Psi : \mathbb{R}^n \to \widehat{\mathbb{R}}^n_{\mathrm{disc}}$ with dense image and such that

$$\chi'_\gamma(\Psi(\xi)) = \chi_\gamma(\xi)$$

for all $(\gamma, \xi) \in \Gamma \times \mathbb{R}^n$. The key point is that the L_∞ norms coincide on trigonometric polynomials. More concretely, let Λ be a finite subset of Γ and consider $f = \sum_{\gamma \in \Lambda} a_\gamma \chi_\gamma$ and $f' = \sum_{\gamma \in \Lambda} a_\gamma \chi'_\gamma$. Then continuity of f, f' and density of $\Psi(\mathbb{R}^n)$ give

$$\|f'\|_{L_\infty(\widehat{\mathbb{R}}^n_{\mathrm{disc}})} \ = \ \sup_{\xi \in \mathbb{R}^n} |f' \circ \Psi(\xi)| \ = \ \sup_{\xi \in \mathbb{R}^n} |f(\xi)| \ = \ \|f\|_{L_\infty(\mathbb{R}^n)}.$$

On the other hand, since the algebra of trigonometric polynomials is preserved in both cases by the heat semigroup and any other Fourier multiplier, we also find that both BMO norms coincide via Ψ

$$\begin{aligned} \|f'\|_{\mathrm{BMO}_{\mathcal{S}}(\widehat{\mathbb{R}}^n_{\mathrm{disc}})} &= \sup_{t \ge 0} \big\| S_t|f'|^2 - |S_t f'|^2 \big\|^{\frac12}_{L_\infty(\widehat{\mathbb{R}}^n_{\mathrm{disc}})} \\ &= \sup_{t \ge 0} \big\| S_t|f|^2 - |S_t f|^2 \big\|^{\frac12}_{L_\infty(\mathbb{R}^n)} \ = \ \|f\|_{\mathrm{BMO}_{\mathcal{S}}(\mathbb{R}^n)}. \end{aligned}$$

Thus, if $\mathcal{A}_{\mathbb{R}^n}$ and $\mathcal{A}_{\widehat{\mathbb{R}}^n_{\mathrm{disc}}}$ are the subalgebras of trigonometric polynomials

$$T_{\widetilde{m}_1} : \mathcal{A}_{\mathbb{R}^n} \to \mathrm{BMO}_{\mathcal{S}}(\mathbb{R}^n) \ \Leftrightarrow \ T_{\widetilde{m}_1} : \mathcal{A}_{\widehat{\mathbb{R}}^n_{\mathrm{disc}}} \to \mathrm{BMO}_{\mathcal{S}}(\widehat{\mathbb{R}}^n_{\mathrm{disc}}).$$

The full equivalence then follows from the weak density of these subalgebras in their respective L_∞ spaces. If we take crossed products with G the same argument works since $\alpha_1(\mathcal{A}_\dagger) \subset \mathcal{A}_\dagger \quad \text{for} \quad \dagger = \mathbb{R}^n$ or $\widehat{\mathbb{R}}^n_{\mathrm{disc}}$, so that $\mathcal{A}_\dagger \rtimes \mathrm{G}$ is a weak-$*$ dense subalgebra. This reduces the problem to the algebras $\mathcal{A}_\dagger \rtimes \mathrm{G}$, for which the same argument above applies.

E. *Smoothness of the lifting multipliers.* Here the smoothness conditions come into play. Indeed, according to Lemma 2.3 ii) we know that our assumptions on $\widetilde{m}_1$ imply that its Fourier inverse transform $k_{\widetilde{m}_1}$ defines a cb-map $T_{k_{\widetilde{m}_1}}$ from $L_\infty(\mathbb{R}^n; \mathcal{B}(\ell_2(\mathrm{G})))$ to $\mathrm{BMO}_c(\mathbb{R}^n; \mathcal{B}(\ell_2(\mathrm{G})))$. However, arguing as in the proof of Lemma 1.6, we conclude that $T_\rtimes : L_\infty(\mathbb{R}^n) \rtimes \mathrm{G} \to \mathrm{BMO}^c_{\mathcal{S}_\rtimes}$ is bounded. □

Remark 2.5. The same hypotheses of Theorem 2.4 imply in fact that the Fourier multiplier T_m is completely bounded as it is checked following the same argument.

The drawback is that we need to find two lifting multipliers for the left and right cocycles. To simplify these conditions, we begin with Theorem A —stated in the Introduction only for left cocycles— showing that for general discrete groups we may work with one lifting multiplier under stronger smoothness conditions.

Proof of Theorem A. If we set

$$\widetilde{m}^\delta(\xi) = \widetilde{m}(\xi)|\xi|^\delta \quad \text{for} \quad \xi \neq 0$$

and $\widetilde{m}^\delta(0) = 0$ with $\delta = \pm\varepsilon$, we find $|\partial_\xi^\beta \widetilde{m}^{\pm\varepsilon}(\xi)| \le c_n |\xi|^{-|\beta|}$ for all $0 \le |\beta| \le [\frac{n}{2}] + 1$ by the chain rule and our hypotheses. In particular, letting $m^{\pm\varepsilon} = \widetilde{m}^{\pm\varepsilon} \circ b_\psi$ we may follow the proof of Theorem 2.4 to show that $T_{m^{\pm\varepsilon}} : \mathcal{L}(\mathrm{G}) \to \mathrm{BMO}^c_{\mathcal{S}_\psi}$ when

b_ψ is a left cocycle and $T_{m^{\pm\varepsilon}} : \mathcal{L}(\mathrm{G}) \to \mathrm{BMO}^r_{\mathcal{S}_\psi}$ when b_ψ is a right cocycle. In fact these maps are cb-bounded, as it follows from Remark 2.5. On the other hand, we recall from [26] that

$$\begin{aligned}\big[\mathrm{BMO}^r_{\mathcal{S}_\psi}, L_2^\circ(\widehat{\mathbb{G}})\big]_{2/p} &= H_p^r(\mathcal{S}_\psi),\\ \big[\mathrm{BMO}^c_{\mathcal{S}_\psi}, L_2^\circ(\widehat{\mathbb{G}})\big]_{2/p} &= H_p^c(\mathcal{S}_\psi),\end{aligned}$$

see [25] for the definition of the Hardy spaces $H_p^r(\mathcal{S}_\psi)$ and $H_p^c(\mathcal{S}_\psi)$. Arguing as in Theorem 2.4 [Point A], we get $T_{m^{\pm\varepsilon}} = \widetilde{m}^{\pm\varepsilon}(0)E_p + J_pT_{m^{\pm\varepsilon}} = J_pT_{m^{\pm\varepsilon}}$. Therefore we conclude by interpolation that

$$T_{m^{\pm\varepsilon}} : L_p(\widehat{\mathbb{G}}) \xrightarrow{cb} H_p^c(\mathcal{S}_\psi)$$

for $2 < p < \infty$ whenever b_ψ is a left cocycle and we must replace column by row if b_ψ is a right cocycle. At any rate, if $A_\psi(\lambda(g)) = \psi(g)\lambda(g)$ stands for the infinitesimal generator of $\mathcal{S}_\psi$, we know from [23] that

$$\|h\|_p \lesssim_{cb} \big\|A_\psi^{+\gamma}h\big\|_{H_p^c}^{\frac12}\big\|A_\psi^{-\gamma}h\big\|_{H_p^c}^{\frac12}$$

for all $\gamma > 0$ and $h \in L_p^\circ(\widehat{\mathbb{G}})$. Taking $\gamma = \varepsilon/2$ and $h = J_pT_mf$, we see that

$$\begin{aligned}\|T_mf\|_p &\leq_{cb} |m_e|\|E_pf\|_p + \|J_pT_mf\|_p\\ &\lesssim_{cb} |m_e|\|f\|_p + \|A_\psi^{+\gamma}h\|_{H_p^c}^{\frac12}\|A_\psi^{-\gamma}h\|_{H_p^c}^{\frac12}\\ &= |m_e|\|f\|_p + \|T_{m^{+\varepsilon}}f\|_{H_p^c}^{\frac12}\|T_{m^{-\varepsilon}}f\|_{H_p^c}^{\frac12} \lesssim_{cb} \|f\|_p.\end{aligned}$$

The L_p cb-boundedness for $1 < p < 2$ follows by duality as in Theorem 2.4. □

Remark 2.6. If ψ is bounded in G it suffices to know that $\big|\partial^\beta\widetilde{m}(\xi)\big| \leq c_n|\xi|^{-|\beta|+\varepsilon}$ for all $|\beta| \leq [\frac{n}{2}]+1$. If ψ^{-1} is bounded in $\mathrm{G}\setminus\mathrm{G}_0$, we just need to control by $|\xi|^{-|\beta|-\varepsilon}$ for the same β's. The first condition holds for inner cocycles and the second for well-separated ones. The argument is very similar to the proof of Theorem A.

Remark 2.7. Applying Lemma 1.6 in full generality we obtain sufficient conditions for the L_p-boundedness of operator-valued Fourier multipliers over discrete groups.

Proof of Theorem B. As in Theorem 2.4, the L_p-boundedness reduces to the $L_\infty \to \mathrm{BMO}$ boundedness. Assume first that $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$ is a left cocycle, then the argument in Theorem 2.4 gives that $T_m : \mathcal{L}(\mathrm{G}) \to \mathrm{BMO}^c_{\mathcal{S}_\psi}$ is bounded. Let us now consider the row case. One more time following our proof above, this is a matter of showing that $T_\rtimes^\dagger : L_\infty(\mathbb{R}^n) \rtimes \mathrm{G} \to \mathrm{BMO}^c_{\mathcal{S}_\rtimes}$ where $T_\rtimes = T_{\widetilde{m}} \rtimes id_{\mathrm{G}}$. As we already noticed in Paragraph 1.4, we have

$$T_\rtimes^\dagger\Big(\sum\nolimits_g f_g\lambda(g)\Big) = \sum\nolimits_g \alpha_{\psi,g}T_{\widetilde{m}}^\dagger\alpha_{\psi,g^{-1}}(f_g)\lambda(g) = \sum\nolimits_g \Pi_g(f_g)\lambda(g)$$

and $j(\sum_g \Pi_g(f_g)\lambda(g)) = \big(\alpha_{\psi,h^{-1}}T_{\widetilde{m}}^\dagger\alpha_{\psi,h}\big)\bullet j(\sum_g f_g\lambda(g)) = \Phi\big(j(\sum_g f_g\lambda(g))\big)$, where

$$\alpha_{\psi,h^{-1}}T_{\widetilde{m}}^\dagger\alpha_{\psi,h}f(x) = \Pi_{h^{-1}}f(x) = \int_{\mathbb{R}^n} \overline{k}_{\widetilde{m}}(\beta_hx - \beta_hy)f(y)\,dy$$

with $\beta_h = \alpha_{\psi,h^{-1}}$ and $\widehat{k}_{\widetilde{m}} = \widetilde{m}$. In particular, $T_\rtimes^\dagger : L_\infty(\mathbb{R}^n) \rtimes \mathrm{G} \to \mathrm{BMO}^c_{\mathcal{S}_\rtimes}$ will be bounded if the conditions in Lemma 1.6 hold. In fact, since the Schur product defining Φ is constant in rows, we may argue as for the proof of Lemma 1.7 with

$\mathcal{M} = \mathbb{C}$ and apply Lemma 2.3 i) to conclude that our smoothness condition is strong enough to imply that of Lemma 1.6. Thus, it remains to check the L_2-boundedness condition

$$\Big\| \Big(\int_{\mathbb{R}^n} \Big| \Big(\Pi_{h^{-1}} f_{gh}(x) \Big) \Big|^2 dx \Big)^{\frac{1}{2}} \Big\|_{\mathcal{B}(\ell_2(\mathrm{G}))} \lesssim \Big\| \Big(\int_{\mathbb{R}^n} \Big| \Big(f_{gh}(x) \Big) \Big|^2 dx \Big)^{\frac{1}{2}} \Big\|_{\mathcal{B}(\ell_2(\mathrm{G}))}.$$

Indeed, the statement of Lemma 1.6 is written in terms of $\Pi_{gh^{-1}}$'s, but a quick look at the proof shows that we may replace them by $\Pi_{h^{-1}}$'s, since we have the identity $\Pi_{h^{-1}} = \alpha_{\psi,g^{-1}} \Pi_{gh^{-1}} \alpha_{\psi,g}$. On the other hand, the cb-inequality follows from the argument below after matrix amplification. Let us thus prove this inequality. Since

$$\widehat{\Pi_{h^{-1}} f}(\xi) = \widehat{\alpha_{\psi,h^{-1}} \overline{k}}_{\widetilde{m}}(\xi) \widehat{f}(\xi) = \overline{\widetilde{m}(-\beta_h \xi)} \widehat{f}(\xi),$$

by Fubini and Plancherel theorems we may write the left hand side as

$$\mathrm{LHS}^2 \ = \sup_{\|\gamma\|_{\ell_2(\mathrm{G})} \le 1} \sum\nolimits_g \int_{\mathbb{R}^n} \Big| \sum\nolimits_h \widetilde{m}(-\beta_h \xi) \overline{\widehat{f}_{gh}(\xi) \gamma_h} \Big|^2 d\xi.$$

Since $\widetilde{m}(-\beta_h \xi) = \widetilde{m}(\alpha_{\psi,h^{-1}}(-\xi))$ and we are assuming that $\|\widetilde{m}\|_{schur} < \infty$, there exists a factorization $\widetilde{m}(\alpha_{\psi,h^{-1}}(-\xi)) = \langle A_{-\xi}, B_{h^{-1}} \rangle_{\mathcal{K}}$ and some positive constant c for which

$$\sup_\xi \|A_\xi\|_{\mathcal{K}}, \ \sup_g \|B_g\|_{\mathcal{K}} \le \sqrt{c}.$$

This yields

$$\begin{aligned} \mathrm{LHS}^2 \ &= \sup_{\|\gamma\|_{\ell_2(\mathrm{G})} \le 1} \sum\nolimits_g \int_{\mathbb{R}^n} \Big| \Big\langle A_{-\xi}, \sum\nolimits_h \overline{\widehat{f}_{gh}(\xi) \gamma_h} B_{h^{-1}} \Big\rangle_{\mathcal{K}} \Big|^2 d\xi \\ &\le \ c \sup_{\|\gamma\|_{\ell_2(\mathrm{G})} \le 1} \sum\nolimits_g \int_{\mathbb{R}^n} \sum\nolimits_j \Big| \sum\nolimits_h \overline{\widehat{f}_{gh}(\xi) \gamma_h} B_{h^{-1}}^j \Big|^2 d\xi, \end{aligned}$$

where $B_{h^{-1}}^j$ denotes the j-th component of $B_{h^{-1}}$. Taking $\gamma^j = (\gamma_h \overline{B_{h^{-1}}^j})_{h \in \mathrm{G}}$

$$\begin{aligned} \mathrm{LHS}^2 \ &\le \ c \sup_{\|\gamma\|_{\ell_2(\mathrm{G})} \le 1} \sum\nolimits_j \sum\nolimits_g \int_{\mathbb{R}^n} \Big| \sum\nolimits_h \widehat{f}_{gh}(\xi) \gamma_h^j \Big|^2 d\xi \\ &= \ c \sup_{\|\gamma\|_{\ell_2(\mathrm{G})} \le 1} \sum\nolimits_j \sum\nolimits_g \int_{\mathbb{R}^n} \Big| \sum\nolimits_h f_{gh}(x) \gamma_h^j \Big|^2 dx \\ &= \ c \sup_{\|\gamma\|_{\ell_2(\mathrm{G})} \le 1} \sum\nolimits_j \Big\langle \gamma^j, \int_{\mathbb{R}^n} \Big| \Big(f_{gh}(x) \Big) \Big|^2 dx \, \gamma^j \Big\rangle_{\ell_2(\mathrm{G})} \ \le \ c^2 \ \mathrm{RHS}^2. \end{aligned}$$

This completes the proof for left cocycles. Alternatively, if we deal with a right cocycle $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$ everything is row/column switched. More concretely, this means that the row BMO estimate follows from our argument in Theorem 2.4 and the column BMO requires Lemma 1.6, details are left to the reader. □

2.3. **Noncommutative Riesz transforms.** Let G be a discrete group, let ψ be a length function on it and construct $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$ to be either the left or right cocycle associated to ψ. The *Riesz transform on* $\mathcal{L}(\mathrm{G})$ associated to an element $\eta \in \mathcal{H}_\psi$ is the multiplier

$$R_\eta \Big(\sum_{g \in \mathrm{G}} \widehat{f}(g) \lambda(g) \Big) = -i \sum_{g \in \mathrm{G}} \frac{\langle b_\psi(g), \eta \rangle_\psi}{\sqrt{\psi(g)}} \widehat{f}(g) \lambda(g).$$

Indeed, note that $b_\psi(g)/\sqrt{\psi(g)}$ is just the normalized vector in the direction of $b_\psi(g)$, so that the classical symbol for the Riesz transform $\widetilde{m}_\eta(\xi) = -i\langle \xi, \eta\rangle_\psi / \|\xi\|_\psi$ is a lifting multiplier for R_η. The classical Mihlin condition clearly holds for $\widetilde{m}_\eta$, but it fails the more restrictive condition in Theorem A for $\varepsilon > 0$. Theorem 2.4 imposes alternatively to find another lifting multiplier $\widetilde{m}'_\eta$ so that $\widetilde{m}_\eta \circ b_\psi = \widetilde{m}'_\eta \circ b'_\psi$. We ignore how to find such function in general. The following result is on the contrary a simple consequence of Theorem B.

Corollary 2.8. *Given a discrete group* G*, consider a length function* $\psi : \mathrm{G} \to \mathbb{R}_+$ *and set* $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$ *to be either the left or right cocycle associated to it. Assume that* $\dim \mathcal{H}_\psi < \infty$*, then any operator in the algebra* $\mathcal{R}$ *generated by the Riesz transforms*

$$\mathcal{R} = \mathrm{span}\Big\{ \prod_{\eta \in \Gamma} R_\eta \, \big| \, \Gamma \textit{ finite set in } \mathcal{H}_\psi \Big\}$$

defines a cb-map $\mathcal{L}(\mathrm{G}) \to \mathrm{BMO}_{\mathcal{S}_\psi}$ *and* $L_p(\widehat{\mathbb{G}}) \to L_p(\widehat{\mathbb{G}})$ *for all* $1 < p < \infty$*.*

Proof. Note that $\|\widetilde{m}_1 \widetilde{m}_2\|_{schur} \le \|\widetilde{m}_1\|_{schur} \|\widetilde{m}_2\|_{schur}$ by taking the Hilbertian tensor product $\mathcal{K} = \mathcal{K}_1 \otimes_2 \mathcal{K}_2$. Moreover, according to the chain rule the product $\widetilde{m}_1 \widetilde{m}_2$ satisfies the smoothness conditions whenever $\widetilde{m}_1$ and $\widetilde{m}_2$ do. Therefore, the Fourier multipliers satisfying the hypotheses of Theorem B form an algebra. In particular, it suffices to check the conditions for a single Riesz transform R_η. We have

$$\begin{aligned} \widetilde{m}_\eta(\alpha_{\psi,g}(\xi)) &= -i \frac{\langle \alpha_{\psi,g}(\xi), \eta \rangle_\psi}{\sqrt{\langle \alpha_{\psi,g}(\xi), \alpha_{\psi,g}(\xi) \rangle_\psi}} \\ &= -i \Big\langle \frac{\xi}{\sqrt{\langle \xi, \xi \rangle_\psi}}, \alpha_{\psi,g^{-1}}(\eta) \Big\rangle_\psi \ = \ \big\langle A_\xi, B_g \big\rangle_{\mathbb{R}^n}, \end{aligned}$$

with A_ξ and B_g satisfying the estimates $\sup_{\xi \in \mathbb{R}^n} |A_\xi| = 1$ and $\sup_{g \in \mathrm{G}} |B_g| = |\eta|$. Hence, the assertion follows since the Hörmander smoothness condition holds. □

2.4. **Mild algebraic/geometric assumptions.** We continue our analysis just imposing the existence of one lifting multiplier. Let us prove our assertion —in the Introduction— that the additional $\varepsilon > 0$ in Theorem A can be removed under any of the following alternative assumptions:

i) G is abelian,
ii) $b_\psi(\mathrm{G})$ is a lattice in $\mathbb{R}^n$,
iii) $\alpha_\psi(\mathrm{G})$ is a finite subgroup of $O(n)$,
iv) The multiplier is ψ-radial, i.e. $m_g = h(\psi(g))$.

Proof. If G is abelian, the Hilbert space $\mathcal{H}_\psi$ and the inclusion map $b_\psi : \mathrm{G} \to \mathcal{H}_\psi$ coincide for both left and right cocycles. Thus, the hypotheses of Theorem 2.4 are satisfied and we deduce the first assertion. For radial multipliers, we note that our smoothness condition implies the boundedness of $T_{\widetilde{m}} : L_\infty(\mathbb{R}^n) \to \mathrm{BMO}_{\mathcal{S}}(\mathbb{R}^n)$ where $\widetilde{m} = h \circ |\,|^2$ and $\mathcal{S}$ denotes the heat semigroup on $\mathbb{R}^n$. Indeed, this follows since $\mathrm{BMO}_{\mathcal{S}}$ is isomorphic to the classical BMO space on $\mathbb{R}^n$ up to a constant depending on the dimension. Since radial Fourier multipliers on $\mathbb{R}^n$ are G-equivariant with respect to any isometric action $\alpha : \mathrm{G} \to O(n)$, we may apply Lemma 1.2 with the cocycle action α_ψ and deduce that $T_\rtimes : L_\infty(\mathbb{R}^n) \rtimes \mathrm{G} \to \mathrm{BMO}_{\mathcal{S}_\rtimes}$ is completely bounded with $T_\rtimes = T_{\widetilde{m}} \rtimes id_{\mathrm{G}}$ and $\mathcal{S}_\rtimes = \mathcal{S} \rtimes id_{\mathrm{G}}$. However, as noticed in the proof

of Theorem B, this is all what is really needed. When $\alpha_\psi(\mathrm{G})$ is a finite subgroup of $O(n)$ we use Theorem B. By [63]

$$\big\|\widetilde{m}\big\|_{schur} \ = \inf_{\substack{\widetilde{m}(\alpha_{\psi,g}(\xi)) = \langle A_\xi, B_g \rangle_{\mathcal{K}} \\ \mathcal{K} \text{ Hilbert}}} \Big(\sup_{\xi \in \mathbb{R}^n} \|A_\xi\|_{\mathcal{K}} \sup_{g \in \mathrm{G}} \|B_g\|_{\mathcal{K}} \Big)$$

coincides with the norm of the Schur multiplier

$$\widetilde{m} : \sum_{\xi \in \mathbb{R}^n} \sum_{\gamma \in \alpha_\psi(\mathrm{G})} a_{\xi,\gamma} e_{\xi,\gamma} \mapsto \sum_{\xi \in \mathbb{R}^n} \sum_{\gamma \in \alpha_\psi(\mathrm{G})} \widetilde{m}(\gamma(\xi))\, a_{\xi,\gamma} e_{\xi,\gamma}$$

on $L_2^c(\mathbb{R}^n) \otimes_h \ell_2^r(\alpha_\psi(\mathrm{G}))$. If $\alpha_\psi(\mathrm{G})$ is a finite set, we may factorize $\widetilde{m}$ as

$$\begin{array}{rcl} L_2^c(\mathbb{R}^n) \otimes_h \ell_2^r(\alpha_\psi(\mathrm{G})) & \xrightarrow{id} & L_2^c(\mathbb{R}^n) \otimes_h \ell_2^c(\alpha_\psi(\mathrm{G})) \\ & \xrightarrow{\widetilde{m}} & L_2^c(\mathbb{R}^n) \otimes_h \ell_2^c(\alpha_\psi(\mathrm{G})) \\ & \xrightarrow{id} & L_2^c(\mathbb{R}^n) \otimes_h \ell_2^r(\alpha_\psi(\mathrm{G})), \end{array}$$

which immediately shows that

$$\big\|\widetilde{m}\big\|_{schur} \ \le \ |\alpha_\psi(\mathrm{G})| \sup_{(\xi,g) \in \mathbb{R}^n \times \mathrm{G}} \big|\widetilde{m}(\alpha_{\psi,g}(\xi))\big| \ \le \ |\alpha_\psi(\mathrm{G})| \, \big\|\widetilde{m}\big\|_\infty \ < \ \infty.$$

On the other hand, the smoothness condition in Theorem B is used to ensure

$$\Omega_{\widetilde{m},\alpha_\psi} \ = \ \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \int_{|y|>2|x|} \sup_{g \in \mathrm{G}} \big| k_{\widetilde{m}}(\alpha_{\psi,g} y - \alpha_{\psi,g} x) - k_{\widetilde{m}}(\alpha_{\psi,g} y) \big| \, dy \ < \ \infty.$$

However, if $\Omega_{\widetilde{m}} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \int_{|y|>2|x|} |k_{\widetilde{m}}(y-x) - k_{\widetilde{m}}(y)| \, dy$, it is well-known that

$$\big|\partial_\xi^\beta \widetilde{m}(\xi)\big| \ \le \ c_n |\xi|^{-|\beta|} \text{ for } |\beta| \le \Big[\frac{n}{2}\Big] + 1 \quad \Rightarrow \quad \Omega_{\widetilde{m}} < \infty.$$

In particular, since $\alpha_\psi(\mathrm{G})$ is a finite set we find that

$$\Omega_{\widetilde{m},\alpha_\psi} \ \le \sum_{\alpha_{\psi,g} \in \alpha_\psi(\mathrm{G})} \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \int_{|y|>2|x|} \big| k_{\widetilde{m}}(\alpha_g y - \alpha_g x) - k_{\widetilde{m}}(\alpha_g y) \big| \, dy \ \le \ \big|\alpha_\psi(\mathrm{G})\big| \, \Omega_{\widetilde{m}}$$

is also finite, which proves assertion iii). It remains to study the case when the image $b_\psi(\mathrm{G})$ lives in a lattice Λ_ψ of the Hilbert space $\mathcal{H}_\psi$. If $\dim \mathcal{H}_\psi = n < \infty$, it is a simple observation that $\alpha_\psi(\mathrm{G})$ must be a finite subgroup of $O(n)$, so that ii) follows from iii). Indeed, since there are finitely many orthogonal transformations leaving Λ_ψ invariant, it suffices to see that $\alpha_{\psi,g}(\Lambda_\psi) \subset \Lambda_\psi$ for all $g \in \mathrm{G}$. We may clearly assume that $b_\psi(\mathrm{G})$ generates $\mathcal{H}_\psi$, so that Λ_ψ is the space of linear combinations $\sum_{h \in \mathrm{G}} \gamma_h b_\psi(h)$ with $\gamma_h \in \mathbb{Z}$. Since

$$\alpha_{\psi,g}(b_\psi(h)) = b_\psi(gh) - b_\psi(g),$$

the $\mathbb{Z}$-linear combinations are stable under $\alpha_{\psi,g}$ for all g and the claim follows. $\square$

3. Littlewood-Paley theory

We now prove some square function estimates. The boundedness of new square functions for noncommutative martingale transforms and semicommutative CZO's was recently investigated in [46]. The smoothness assumptions there where needed for additional weak-type $(1,1)$ estimates, here we will find weaker conditions. Set $\mathcal{R}_1 = L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{M}_1$ and $\mathcal{R}_{12} = L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{M}_1\bar{\otimes}\mathcal{M}_2$ with $\mathcal{M}_1$ and $\mathcal{M}_2$ semifinite algebras. Consider the CZO formally given by

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y) \otimes f(y)\, dy = \int_{\mathbb{R}^n} \widetilde{k}(x,y)(f(y))\, dy,$$

where $\widetilde{k}(x,y)(\cdot) = k(x,y) \otimes \cdot$ and k takes values in $\mathcal{M}_2$. If

- $T : L_\infty(\mathcal{M}_1; L_2^c(\mathbb{R}^n)) \to L_\infty(\mathcal{M}_1\bar{\otimes}\mathcal{M}_2; L_2^c(\mathbb{R}^n))$,
- $\displaystyle \operatorname*{ess\,sup}_{x_1,x_2} \int_{|x_1-y|>2|x_1-x_2|} \big\| k(x_1,y) - k(x_2,y) \big\|_{\mathcal{M}_2}\, dy \, < \, \infty$,

we deduce from Lemma 1.3 that $T : \mathcal{R}_1 \to \mathrm{BMO}_{\mathcal{R}_2}^c$. Take $\mathcal{M}_2 = \mathcal{B}(\ell_2)$ and $k(x,y) = \sum_m k_m(x,y) \otimes (e_{1m} \oplus_\infty e_{m1})$, where the k_m's are scalar-valued and e_{ij} stands for the (i,j)-th matrix unit. Consider the CZO T_m associated to k_m and such that $T = \sum_m T_m \otimes (e_{1m} \oplus_\infty e_{m1})$. The column part $T_c = \sum_m T_m \otimes e_{m1}$ satisfies the first condition if $\|T_c f\|_2 = (\sum_m \|T_m f\|_2^2)^{1/2} \lesssim \|f\|_2$ since the kernel acts by left multiplication, see Remark 1.4. For the row part, we use some basic operator space theory [64]. Namely, the condition is equivalent to the cb-boundedness of $T_r : L_2^c(\mathbb{R}^n) \to L_2^c(\mathbb{R}^n) \otimes_h R$. In particular, such a map defines an element in $L_2^c(\mathbb{R}^n)\bar{\otimes} L_2^r(\mathbb{R}^n) \otimes_h R$ with norm $\| \sum_m T_m T_m^* \|^{1/2}$. This leads to the same condition with T_m^* in place of T_m. Finally

$$\operatorname*{ess\,sup}_{x_1,x_2} \int_{|x_1-y|>2|x_1-x_2|} \Big(\sum_{m=1}^\infty \big| k_m(x_1,y) - k_m(x_2,y) \big|^2 \Big)^{\frac12} dy \, < \, \infty$$

is the form of the smoothness assumption. By the symmetry of the kernel, the map $T^\dagger(f) = T(f^*)^*$ essentially equals T and $\mathcal{R}_1 \to \mathrm{BMO}_{\mathcal{R}_{12}}$ boundedness follows with no extra assumptions. L_p-boundedness follows by interpolation and duality if the Hörmander condition holds also on the second variable. In particular, if we let $\mathcal{R} = L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{M}$, we recover the main result in [46] in terms of the spaces

$$L_p(\mathcal{R}; \ell_{rc}^2) = \begin{cases} L_p(\mathcal{R}; \ell_2^r) + L_p(\mathcal{R}; \ell_2^c) & \text{if } 1 \le p \le 2, \\ L_p(\mathcal{R}; \ell_2^r) \cap L_p(\mathcal{R}; \ell_2^c) & \text{if } 2 \le p \le \infty. \end{cases}$$

Lemma 3.1. *Let*

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y) \otimes f(y)\, dy = \sum_{m=1}^\infty T_m f(x) \otimes (e_{1m} \oplus_\infty e_{m1})$$

be a formal expression of the CZO *above. Assume that*

i) $\displaystyle \sum_{m=1}^\infty \|T_m f\|_2^2 + \|T_m^* f\|_2^2 \, \lesssim \, \|f\|_2^2$,

ii) $\displaystyle \operatorname*{ess\,sup}_{x_1,x_2} \int_{|x_1-y|>2|x_1-x_2|} \big\| k(x_1,y) - k(x_2,y) \big\|_{\ell_2} dy \, < \, \infty$,

iii) $\displaystyle \operatorname*{ess\,sup}_{x_1,x_2} \int_{|x_1-y|>2|x_1-x_2|} \big\| k(y,x_1) - k(y,x_2) \big\|_{\ell_2} \, dy \, < \, \infty.$

Then $T: \mathcal{R} \to \mathrm{BMO}_{\mathcal{R}}$ *is bounded and we find for* $1 < p < \infty$

$$\Big\| \sum_{m=1}^{\infty} T_m f \otimes \delta_m \Big\|_{L_p(\mathcal{R};\ell_{rc}^2)} \lesssim \frac{p^2}{p-1} \, \|f\|_{L_p(\mathcal{R})}.$$

This gives the L_p-boundedness of operator-valued g-functions and Lusin square functions, see [46] for more applications. The conditions above hold for convolution maps with kernels satisfying $(\sum_m |\partial_\xi^\beta \, \widehat{k}_m(\xi)|^2)^{\frac12} \, \le \, c_n \, |\xi|^{-|\beta|}$ for $|\beta| \le [\frac{n}{2}]+1$, which is just a form of Hörmander-Mihlin multiplier theorem for ℓ_2-valued kernels.

Lemma 3.2. *Let* $\psi: \mathrm{G} \to \mathbb{R}_+$ *be a length function with* $\dim \mathcal{H}_\psi = n$. *Let* Γ *stand for the free group* $\mathbb{F}_\infty$ *with infinitely many generators* $\gamma_1, \gamma_2, \ldots$ *and left regular representation* λ_Γ. *Consider a sequence of functions* $(h_m)_{m \ge 1}$ *in* $\mathcal{C}^{k_n}(\mathbb{R}_+ \setminus \{0\})$ *for* $k_n = [\frac{n}{2}] + 1$ *such that*

$$\Big(\sum_{m=1}^{\infty} \Big| \frac{d^k}{d\xi^k} h_m(\xi) \Big|^2 \Big)^{\frac12} \, \le \, c_n \, |\xi|^{-k} \quad \textit{for all} \quad k \le \Big[\frac{n}{2} \Big] + 1.$$

Let $k_m: \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ *given by* $\widehat{k}_m(\xi) = h_m(|\xi|^2)$. *Then, we find a cb-map*

$$\Lambda: L_\infty(\mathbb{R}^n) \bar{\otimes} \mathcal{L}(\Gamma) \ni \sum_{\gamma \in \Gamma} f_\gamma \otimes \lambda_\Gamma(\gamma) \mapsto \sum_{m=1}^{\infty} \overline{k}_m * f_{\gamma_m} \in \mathrm{BMO}_{\mathbb{R}^n}.$$

Proof. According to the noncommutative Khintchine inequality for free generators [64], the map $e_{1m} \oplus_\infty e_{m1} \mapsto \lambda_\Gamma(\gamma_m)$ is a cb-isomorphism and the span of $\lambda_\Gamma(\gamma_m)$'s is cb-complemented in $\mathcal{L}(\Gamma)$. In particular, it suffices to show that we have a cb map $L_\infty(\mathbb{R}^n) \bar{\otimes} \mathcal{B}(\ell_2) \ni \sum_m f_m \otimes \big(e_{1m} \oplus_\infty e_{m1} \big) \mapsto \sum_m \overline{k}_m * f_m \in \mathrm{BMO}_{\mathbb{R}^n}$. Since we have an intersection of row and column at both sides, it is enough to prove the row-row and column-column cb-boundedness. By symmetry, we just consider the column case, so we are reduced to show that $\sum_m f_m \otimes e_{m1} \mapsto \sum_m \overline{k}_m * f_m$ defines a cb-map $C(L_\infty(\mathbb{R}^n)) \to \mathrm{BMO}_{\mathbb{R}^n}^c$, where $C(L_\infty(\mathbb{R}^n)) = C \otimes_{\min} L_\infty(\mathbb{R}^n)$ and C stands for the column subspace of $\mathcal{B}(\ell_2)$. It suffices to show the validity of the predual inequality, which easily reduces to

$$\Big\| \Big(\sum_{m=1}^{\infty} \big\| k_m * \varphi \big\|_{\ell_2}^2 \Big)^{\frac12} \Big\|_{L_1(\mathbb{R}^n)} \lesssim \|\varphi\|_{H_1(\ell_2)}$$

for ℓ_2-valued functions. Using the atomic characterization of $H_1(\ell_2)$, we may write its norm as $\inf \sum_k |\lambda_k|$ where the infimum runs over all possible decompositions $\varphi = \sum_k \lambda_k a_k$ as a linear combination of atoms a_k, which are mean zero functions $\mathbb{R}^n \to \ell_2$ supported by cubes and such that $\|a_k\|_{L_2(\ell_2)} \le |\mathrm{supp}\, a_k|^{-1/2}$. By the triangle inequality, it suffices to see that the left hand side is $\lesssim 1$ when φ is an arbitrary atom a supported by an arbitrary cube Q. We have

$$\int_{\mathbb{R}^n} \Big(\sum_{m=1}^{\infty} \big\| k_m * a(x) \big\|_{\ell_2}^2 \Big)^{\frac12} dx \, = \, \int_{5Q} + \int_{\mathbb{R}^n \setminus 5Q} \, = \, \mathrm{A} + \mathrm{B}.$$

Our hypotheses easily give

$$\Big(\sum_{m=1}^{\infty} \big| \partial_\xi^\beta \widehat{k}_m(\xi) \big|^2 \Big)^{\frac{1}{2}} \, \le \, c_n \, |\xi|^{-|\beta|} \quad \text{for all } \beta \text{ such that } \quad |\beta| \le \Big[\frac{n}{2} \Big] + 1.$$

As we remarked before the statement of this result, this implies the hypotheses of Lemma 3.1. In particular, we have $\sum_m \|k_m * f\|_2^2 \lesssim \|f\|_2^2$. This, together with Hölder's inequality gives rise to

$$\begin{array}{rcl} \mathrm{A} & \le & \sqrt{|5Q|} \, \Big(\displaystyle\int_{5Q} \sum_{m=1}^{\infty} \big\| k_m * a(x) \big\|_{\ell_2}^2 \, dx \Big)^{\frac{1}{2}} \\ & \le & \sqrt{|5Q|} \, \Big(\displaystyle\sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \big\| k_m * a_j(x) \big\|_2^2 \Big)^{\frac{1}{2}} \, \lesssim \, \sqrt{|5Q|} \, \Big(\displaystyle\int_{\mathbb{R}^n} \big\| a(x) \big\|_{\ell_2}^2 \, dx \Big)^{\frac{1}{2}} \, \lesssim \, 1, \end{array}$$

for $a = (a_j)_{j \ge 1}$. On the other hand, using the mean-zero condition

$$\begin{array}{rcl} \mathrm{B} & = & \displaystyle\int_{\mathbb{R}^n \setminus 5Q} \Big(\sum_{m=1}^{\infty} \Big\| \int_Q \big(k_m(x-y) - k_m(x-c_Q) \big) a(y) \, dy \Big\|_{\ell_2}^2 \Big)^{\frac{1}{2}} \, dx \\ & \le & \displaystyle\int_Q \Big[\int_{\mathbb{R}^n \setminus 5Q} \Big(\sum_{m=1}^{\infty} \big| k_m(x-y) - k_m(x-c_Q) \big|^2 \Big)^{\frac{1}{2}} \, dx \Big] \, \big\| a(y) \big\|_{\ell_2} \, dy \\ & \lesssim & \displaystyle\int_Q \big\| a(y) \big\|_{\ell_2} \, dy \, \le \, \sqrt{|Q|} \, \Big(\int_{\mathbb{R}^n} \big\| a(y) \big\|_{\ell_2}^2 \, dy \Big)^{\frac{1}{2}} \, \le \, 1, \end{array}$$

according to condition iii) in Lemma 3.1, which holds as a consequence of the Hörmander-Mihlin condition in the statement. The estimates for A and B show that the predual inequality holds and the proof is complete. $\square$

Theorem 3.3. *Let*

$$Tf = \sum\nolimits_m T_m f \otimes \lambda_\Gamma(\gamma_m) = \sum\nolimits_{g,m} h_m(\psi(g)) \widehat{f}(g) \lambda(g) \otimes \lambda_\Gamma(\gamma_m)$$

for $f = \sum_g \widehat{f}(g) \lambda(g)$. *Then, the following square function inequalities hold*:

i) *If* $\mathrm{G}_\Gamma = \mathrm{G} \times \Gamma$, *the maps*

$$T : \mathcal{L}(\mathrm{G}) \to \mathrm{BMO}_{\mathcal{S}_{\psi,\otimes}}(\mathcal{L}(\mathrm{G}_\Gamma)) \quad \textit{and} \quad T : L_p(\widehat{\mathbb{G}}) \to L_p(\widehat{\mathbb{G}}_\Gamma)$$

are completely bounded for $1 < p < \infty$, *where* $\mathcal{S}_{\psi,\otimes} = (S_{\psi,t} \otimes id_{\mathcal{L}(\Gamma)})_{t \ge 0}$. *In particular, we find the square function inequalities*

$$\Big\| \sum_{m=1}^{\infty} T_m f \otimes \delta_m \Big\|_{L_p(\widehat{\mathbb{G}}; \ell_{rc}^2)} \, \le_{cb} \, c_p \, \|f\|_{L_p(\widehat{\mathbb{G}})},$$

where $L_p(\widehat{\mathbb{G}}; \ell_{rc}^2)$ *refers to the space* $L_p(\mathcal{R}; \ell_{rc}^2)$ *defined above for* $\mathcal{R} = \mathcal{L}(\mathrm{G})$.

ii) *Additionally, we have*

$$\sum_{m=1}^{\infty} |h_m(\xi)|^2 = 1 \quad \Rightarrow \quad \|f\|_{L_p(\widehat{\mathbb{G}})} \, \le_{cb} \, c_p \, \Big\| \sum_{m=1}^{\infty} T_m f \otimes \delta_m \Big\|_{L_p(\widehat{\mathbb{G}}; \ell_{rc}^2)}.$$

Proof. As it follows from our proof of Lemma 3.2, the smoothness conditions that we have imposed on the h_m's, together with Lemma 3.1 and the fact that $e_{1m} \oplus_\infty e_{m1} \mapsto \lambda_\Gamma(\gamma_m)$ is a cb-isomorphism, imply that we have a cb-map

$$T' : \sum_{g \in \mathrm{G}} f_g \lambda(g) \in \mathcal{R}_1 \mapsto \sum_{m=1}^\infty \sum_{g \in \mathrm{G}} (k_m * f_g) \lambda(g) \otimes \lambda_\Gamma(\gamma_m) \in \mathrm{BMO}_{\mathcal{R}_{12}},$$

where $\mathcal{R}_1 = L_\infty(\mathbb{R}^n) \rtimes \mathrm{G}$ and $\mathcal{R}_{12} = \mathcal{R}_1 \bar{\otimes} \mathcal{L}(\Gamma)$. The BMO space is cb-isomorphic to $\mathrm{BMO}_{\mathcal{S}_\otimes}$ given by $\mathcal{S}_\otimes = (S_t \rtimes id_{\mathcal{L}(\mathrm{G}_\Gamma)})_{t \geq 0}$. As in Theorem 2.4, we may replace $\mathbb{R}^n$ by its Bohr compactification (which in turn is isometric to $\mathcal{H}_\psi$) and use the embedding $\pi_\psi : \mathcal{L}(\mathrm{G}) \to \mathcal{L}(\mathcal{H}_\psi) \rtimes \mathrm{G}$ to obtain that $T' \circ \pi_\psi = (\pi_\psi \otimes id_{\mathcal{L}(\Gamma)}) \circ T$. Here it is relevant to recall the identity $k_m * \exp b_\psi(g) = h_m(\psi(g)) \exp b_\psi(g)$, which follows from

$$\begin{aligned} k_m * \exp b_\psi(g) &= \int_{\mathbb{R}^n} k_m(\,\cdot - y)\, e^{2\pi i \langle b_\psi(g), y\rangle}\, dy \\ &= \int_{\mathbb{R}^n} k_m(y) e^{-2\pi i \langle b_\psi(g), y\rangle}\, dy\; e^{2\pi i \langle b_\psi(g), \cdot\rangle} \;=\; h_m(\psi(g)) \exp b_\psi(g). \end{aligned}$$

Arguing as in Theorem 2.4, we get $T : \mathcal{L}(\mathrm{G}) \xrightarrow{cb} \mathrm{BMO}_{\mathcal{S}_{\psi,\otimes}}(\mathcal{L}(\mathrm{G}_\Gamma))$. The complete boundedness on L_2 follows immediately from the smoothness condition for $k = 0$ on the h_m's. Thus, by the usual interpolation argument we obtain the cb-boundedness for $2 < p < \infty$. The case $1 < p < 2$ is slightly different because T is not self-dual as in Theorem 2.4. We have $T^*(\sum_{\gamma \in \Gamma} f_\gamma \otimes \lambda_\Gamma(\gamma)) = \sum_m T_m^*(f_{\gamma_m})$. Since the L_2 boundedness is clear, it suffices to show that $T^* : \mathcal{L}(\mathrm{G}_\Gamma) \to \mathrm{BMO}_{\mathcal{S}_\psi}$ is completely bounded. Indeed, arguing once more as in Theorem 2.4 we find

$$T^*\Big(\sum_{\gamma \in \Gamma} f_\gamma \otimes \lambda_\Gamma(\gamma) \Big) = J_p T^* + \sum_{g \in \mathrm{G}_0} \sum_{m=1}^\infty \overline{h_m(0)}\, \widehat{f}_{\gamma_m}(g) \lambda(g),$$

where $J_p : L_p(\widehat{\mathbb{G}}) \to L_p^\circ(\widehat{\mathbb{G}})$ and $\mathrm{G}_0 = \{ g \in \mathrm{G} \,|\, \psi(g) = 0\}$. The first term on the right is cb-bounded on L_p by interpolation. To estimate the L_p-norm of the second term we use Cauchy-Schwartz, the conditional expectation $\mathcal{E}_0$ onto the closure of $\operatorname{span} \lambda(\mathrm{G}_0)$, the noncommutative Khintchine inequality for free generators and the fact that the span of the $\lambda_\Gamma(\gamma_m)$'s is completely complemented in $\mathcal{L}(\Gamma)$, see e.g. [56, 64]. Altogether gives rise to the following estimate

$$\begin{aligned} \Big\| \sum_{g \in \mathrm{G}_0} \sum_{m=1}^\infty \overline{h_m(0)}\, \widehat{f}_{\gamma_m}(g) \lambda(g) \Big\|_p &\leq_{cb} \Big(\sum_{m=1}^\infty |h_m(0)|^2 \Big)^{\frac12} \Big\| \Big(\sum_{m=1}^\infty |\mathcal{E}_0(f_{\gamma_m})|^2 \Big)^{\frac12} \Big\|_p \\ &\leq_{cb} \Big(\sum_{m=1}^\infty |h_m(0)|^2 \Big)^{\frac12} \Big\| \Big(\sum_{m=1}^\infty |f_{\gamma_m}|^2 \Big)^{\frac12} \Big\|_p \\ &\lesssim_{cb} \Big(\sum_{m=1}^\infty |h_m(0)|^2 \Big)^{\frac12} \Big\| \sum_{m=1}^\infty f_{\gamma_m} \otimes \lambda_\Gamma(\gamma_m) \Big\|_p \\ &\leq_{cb} \Big(\sum_{m=1}^\infty |h_m(0)|^2 \Big)^{\frac12} \Big\| \sum_{\gamma \in \Gamma} f_\gamma \otimes \lambda_\Gamma(\gamma) \Big\|_p. \end{aligned}$$

This proves the claim. For the $L_\infty \to \mathrm{BMO}$ estimate, we recall that

$$\Lambda : L_\infty(\mathbb{R}^n)\bar{\otimes}\mathcal{L}(\Gamma) \ni \sum_{\gamma \in \Gamma} f_\gamma \otimes \lambda_\Gamma(\gamma) \mapsto \sum_{m=1}^\infty \overline{k}_m * f_{\gamma_m} \in \mathrm{BMO}_{\mathbb{R}^n}$$

is cb-bounded from Lemma 3.2. Replacing again $\mathbb{R}^n$ by its Bohr compactification and $\mathrm{BMO}_{\mathbb{R}^n}$ by $\mathrm{BMO}_\mathcal{S}$, we use that Λ is G-equivariant (recall that k_m is radial) with respect to the natural action α_ψ and apply Lemma 1.2. This shows that $\Lambda \rtimes id_\mathrm{G} : \mathcal{R}_{12} \to \mathrm{BMO}_{\mathcal{S}_\rtimes}(\mathcal{R}_1)$ is cb-bounded. Finally, we observe that $\Lambda \rtimes id_\mathrm{G} = T'^*$ and the intertwining identity $T'^* \circ (\pi_\psi \otimes id_{\mathcal{L}(\Gamma)}) = \pi_\psi \circ T^*$ still holds. Hence, we get that $T^* : \mathcal{L}(\mathrm{G}_\Gamma) \to \mathrm{BMO}_{\mathcal{S}_\psi}$ is a cb-map and T is cb-bounded on L_p for $1 < p < \infty$. Thus, we conclude

$$\Big\| \sum_{m=1}^\infty T_m f \otimes \delta_m \Big\|_{L_p(\widehat{\mathbb{G}}; \ell_{rc}^2)} \le_{cb} c_p \, \|f\|_{L_p(\widehat{\mathbb{G}})}$$

according to the noncommutative Khintchine inequality for free generators. The proof of ii) is straightforward. Indeed, if $\sum_m |h_m(\xi)|^2 = 1$ it is clear that we find an isometry $\|Tf\|_2 = \|f\|_2$. By polarization, we obtain $\langle f_1, f_2 \rangle_{L_2(\widehat{\mathbb{G}})} = \langle Tf_1, Tf_2 \rangle_{L_2(\widehat{\mathbb{G}}_\Gamma)}$. Therefore, if $f \in L_2(\widehat{\mathbb{G}}) \cap L_p(\widehat{\mathbb{G}})$ we see that

$$\|f\|_p \, = \, \sup \Big\{ \big\langle Tf, Tg \big\rangle_{L_2(\widehat{\mathbb{G}}_\Gamma)} \, \big| \, g \in L_2(\widehat{\mathbb{G}}) \cap L_{p'}(\widehat{\mathbb{G}}), \ \|g\|_{p'} \le 1 \Big\} \, \lesssim \, \|Tf\|_p.$$

By density, this inequality still holds in the whole $L_p(\widehat{\mathbb{G}})$. Moreover, the same estimate is valid after matrix amplification and we deduce the assertion once more by means of the noncommutative Khintchine inequality for free generators. □

Remark 3.4. There exists an alternative formulation of Theorem 3.3 which is also standard in classical Littlewood-Paley theory. Namely, let $\rho : \mathbb{R}^n \to \mathbb{C}$ be a radial function in the Schwartz class $\mathcal{S}_{\mathbb{R}^n}$ and assume that $\sum_{m \in \mathbb{Z}} |\rho(2^{-m}\xi)|^2$ is uniformly bounded on $\xi \in \mathbb{R}^n$. Then, Theorem 3.3 also holds for the functions $h_m(\xi) = \rho(2^{-m}\sqrt{\xi}, 0, \ldots, 0)$. We may also provide Littlewood-Paley type estimates associated to non-radial Fourier multipliers by means of Lemma 3.1 and our results for nonequivariant CZO's.

4. Dimension free estimates

Our results so far imposed $\dim \mathcal{H}_\psi < \infty$ for the tangent space associated to our length function ψ. We are now interested on L_p estimates for Fourier multipliers exploiting the structure of infinite-dimensional cocycles on G. This will include dimension free estimates for noncommutative Riesz transforms —both in gradient and cocycle forms— and radial Fourier multipliers.

4.1. **Khintchine type inequalities.** Our results on Riesz transforms will rely on Pisier's method [60] and a modified version of the noncommutative Khintchine inequality, which goes back to [38, 39]. Given a noncommutative measure space $(\mathcal{M}, \tau)$, we set $RC_p(\mathcal{M})$ as the space of sequences in $L_p(\mathcal{M})$ equipped with the norm

$$\big\| (f_k) \big\|_{RC_p(\mathcal{M})} = \begin{cases} \displaystyle \inf_{f_k = g_k + h_k} \Big\| \big(\sum_k g_k^* g_k \big)^{\frac12} \Big\|_p + \Big\| \big(\sum_k h_k h_k^* \big)^{\frac12} \Big\|_p & \text{if } 1 \le p \le 2, \\ \max \Big\{ \Big\| \big(\sum_k f_k^* f_k \big)^{\frac12} \Big\|_p , \, \Big\| \big(\sum_k f_k f_k^* \big)^{\frac12} \Big\|_p \Big\} & \text{if } 2 \le p \le \infty. \end{cases}$$

The noncommutative Khintchine inequality reads as $G_p(\mathcal{M}) = RC_p(\mathcal{M})$, where $G_p(\mathcal{M})$ denotes the closed span in $L_p(\Omega,\mu;L_p(\mathcal{M}))$ of a family (γ_k) of centered independent Gaussian variables in (Ω,μ). The specific statement for $1 \le p < \infty$ is

$$\Big(\int_\Omega \Big\| \sum\nolimits_k \gamma_k(w) f_k \Big\|_{L_p(\mathcal{M})}^p \, d\mu(w) \Big)^{\frac{1}{p}} \sim_{c_p} \big\| (f_k) \big\|_{RC_p(\mathcal{M})}.$$

Our goal is to prove a similar result adding a group action to the picture. Let $\mathcal{H}$ be a separable real Hilbert space. Choosing a orthonormal basis (e_k), we consider the linear map $B : \mathcal{H} \to L_2(\Omega,\mu)$ given by $B(h) = \sum_k \langle h, e_k \rangle_{\mathcal{H}} \, \gamma_k$. Let Σ stand for smallest σ-algebra making all the $B(h)$'s measurable. Then the well-known gaussian measure space construction [4] tells us that, for every real unitary $\sigma \in O(\mathcal{H})$, we can construct a measure preserving automorphism α_σ on $L_\infty(\Omega,\Sigma,\mu)$ such that $\alpha_\sigma(B(h)) = B(\sigma(h))$. Now, assume that a discrete group G acts by real unitaries on $\mathcal{H}$ and isometrically on some finite von Neumann algebra $\mathcal{M}$. Set

$$G_p(\mathcal{M}) \rtimes \mathrm{G} \, = \, \Big\{ \sum_{h \in \mathcal{H}} \sum_{g \in \mathrm{G}} (B(h) \otimes f_{g,h}) \lambda(g) \Big\} \, \subset \, L_p\big(L_\infty(\Omega,\Sigma,\mu;\mathcal{M}) \rtimes \mathrm{G} \big).$$

We will also need the conditional expectation $E(\sum_g f_g \lambda(g)) = \sum_g (\int_\Omega f_g d\mu) \lambda(g)$, which takes $L_p(L_\infty(\Omega,\Sigma,\mu) \bar{\otimes} \mathcal{M}) \rtimes \mathrm{G})$ to $L_p(\mathcal{M} \rtimes \mathrm{G})$. The conditional L_p norms

$$L_p^{rc}(E) \, = \, \begin{cases} L_p^r(E) + L_p^c(E) & \text{if } 1 \le p \le 2, \\ L_p^r(E) \cap L_p^c(E) & \text{if } 2 \le p \le \infty. \end{cases}$$

are determined by $\|f\|_{L_p(E)^r} = \|E(ff^*)^{\frac12}\|_p$ and $\|f\|_{L_p(E)^c} = \|E(f^*f)^{\frac12}\|_p$ —the row/column conditional spaces— see e.g. [30] for the slight modifications needed for $1 \le p < 2$. Define $RC_p(\mathcal{M}) \rtimes \mathrm{G}$ as the gaussian space $G_p(\mathcal{M}) \rtimes \mathrm{G}$, with the norm inherited from $L_p^{rc}(E)$.

Lemma 4.1. *If $2 \le p < \infty$, we find*

$$\|f\|_{RC_p(\mathcal{M}) \rtimes \mathrm{G}} \le \|f\|_{G_p(\mathcal{M}) \rtimes \mathrm{G}} \le C(p) \|f\|_{RC_p(\mathcal{M}) \rtimes \mathrm{G}}.$$

Proof. The first inequality follows trivially from the continuity of the conditional expectation on $L_{p/2}$. The proof of the second one relies on a suitable application of the central limit theorem. Indeed, assume that f is given by a finite sum of the form $f = \sum_{h,g} (B(h) \otimes f_{g,h}) \lambda(g)$. If we fix $m \ge 1$, we may use the diagonal action on $\ell_2^m(\mathcal{H})$ and repeat the gaussian measure space construction on the larger Hilbert space, resulting in a map $\ell_2^m(\mathcal{H}) \to L_2(\Omega_m,\Sigma_m,\mu_m)$. Let $\phi_m : \mathcal{H} \to \ell_2^m(\mathcal{H})$ denote the isometric diagonal embedding $h \mapsto \frac{1}{\sqrt{m}} \sum_j h \otimes e_j$ and $F_1, F_2, \ldots, F_k$ be bounded measurable functions. Then

$$\pi\big(F_1(B(h_1)) \cdots F_k(B(h_k)) \big) \, = \, F_1(B(\phi_m(h_1)) \cdots F_k(B(\phi_m(h_k))$$

extends to a measure preserving $*$-homomorphism $L_\infty(\Omega,\Sigma,\mu) \to L_\infty(\Omega_m,\Sigma_m,\mu_m)$ which is in addition G-equivariant, i.e. $\pi(\alpha_g(f)) = \alpha_g(\pi(f))$. Thus, we obtain a trace preserving isomorphism $\pi_{\mathrm{G}} = (\pi \otimes id_{\mathcal{M}}) \rtimes id_{\mathrm{G}}$ from $L_\infty(\Omega,\Sigma,\mu;\mathcal{M}) \rtimes \mathrm{G}$ to the larger space $L_\infty(\Omega_m,\Sigma_m,\mu_m;\mathcal{M}) \rtimes \mathrm{G}$. This implies

$$\|f\|_{G_p(\mathcal{M}) \rtimes \mathrm{G}} \, = \, \Big\| \frac{1}{\sqrt{m}} \sum_{j=1}^m \sum_{h,g} \big(B(h \otimes e_j) \otimes f_{g,h} \big) \lambda(g) \Big\|_p.$$

We may define the random variables $f_j = \sum_{h,g} (B(h \otimes e_j) \otimes f_{g,h})\lambda(g)$, which are independent over E, see [33, 35] for details. Hence, the noncommutative Rosenthal inequality applies and we have

$$\|f\|_{G_p(\mathcal{M})\rtimes \mathrm{G}} \le \frac{Cp}{\sqrt{m}} \Big[\Big(\sum_{j=1}^m \|f_j\|_p^p \Big)^{\frac{1}{p}} + \Big\| \Big(\sum_{j=1}^m E(f_j^* f_j) \Big)^{\frac12} \Big\|_p + \Big\| \Big(\sum_{j=1}^m E(f_j f_j^*) \Big) \Big)^{\frac12} \Big\|_p \Big].$$

Note that $E(f_j f_j^*) = E(ff^*)$ and $E(f_j^* f_j) = E(f^* f)$ for all j. Moreover, we also have $\|f_j\|_p = \|f\|_p$. Therefore, the second inequality follows sending $m \to \infty$. $\square$

Remark 4.2. An improved Rosenthal's inequality [36] actually yields

$$\|f\|_{G_p(\mathcal{M})\rtimes \mathrm{G}} \le C\sqrt{p}\, \|f\|_{RC_p(\mathcal{M})\rtimes \mathrm{G}},$$

which will provide the correct order of the constant in our Khintchine inequality.

Theorem 4.3. *If $1 < p < \infty$, we find*

$$C_1 \sqrt{\frac{p-1}{p}} \|f\|_{RC_p(\mathcal{M})\rtimes \mathrm{G}} \ \le\ \|f\|_{G_p(\mathcal{M})\rtimes \mathrm{G}} \ \le\ C_2 \sqrt{p} \|f\|_{RC_p(\mathcal{M})\rtimes \mathrm{G}}.$$

$G_p(\mathcal{M}) \rtimes \mathrm{G}$ is complemented in $L_p\big(L_\infty(\Omega, \Sigma, \mu; \mathcal{M}) \rtimes \mathrm{G}\big)$ with $c_p \sim \sqrt{p^2/p-1}$.

Proof. The gaussian projection is given by

$$Q(f) = \sum\nolimits_k \Big(\int_\Omega f \gamma_k \, d\mu \Big) \gamma_k,$$

which is independent of the choice of the basis. Let $\widehat{Q} = (Q \otimes id_{\mathcal{M}}) \rtimes id_{\mathrm{G}}$ be the amplified gaussian projection on $L_p\big(L_\infty(\Omega, \Sigma, \mu; \mathcal{M}) \rtimes \mathrm{G}\big)$. It is clear that $G_p(\mathcal{M}) \rtimes \mathrm{G}$ is the image of this L_p space under the gaussian projection. Similarly $RC_p(\mathcal{M}) \rtimes \mathrm{G}$ is the image of $L_p^{rc}(E)$. Note that

$$\widehat{Q} : L_p^{rc}(E) \to RC_p(\mathcal{M}) \rtimes \mathrm{G}$$

is a contraction. Indeed, $E(ff^*) = E(\widehat{Q}f\widehat{Q}f^*) + E(\widehat{Q}^\perp f \widehat{Q}^\perp f^*) \ge E(\widehat{Q}f\widehat{Q}f^*)$ by orthogonality and the same holds for the column case. This immediately gives that

$$\|f\|_{RC_p(\mathcal{M})\rtimes \mathrm{G}} = \sup_{\|g\|_{RC_q} \le 1} \big| \mathrm{tr}(fg) \big| \le \Big(\sup_{\|g\|_{RC_q} \le 1} \|g\|_{G_q(\mathcal{M})\rtimes \mathrm{G}} \Big) \|f\|_{G_p(\mathcal{M})\rtimes \mathrm{G}}$$

with $1/p + 1/q = 1$. Let us now prove the statement. According to Lemma 4.1 and Remark 4.2, we know the first assertion holds for $p \ge 2$. The first inequality for $p \le 2$ now follows from the duality estimate above. The second one is a consequence of the continuous inclusion $L_p^{rc}(E) \to L_p$ for $p \le 2$, see [33, Theorem 7.1]. It remains to prove the complementation result. Since the gaussian projection is self-adjoint we may assume $p \le 2$ and

$$\begin{aligned} \big\| \widehat{Q}f \big\|_{G_p(\mathcal{M})\rtimes \mathrm{G}} &\le C_1 \big\| \widehat{Q}f \big\|_{RC_p(\mathcal{M})\rtimes \mathrm{G}} \\ &= C_1 \sup_{\|g\|_q \le 1} \big| \mathrm{tr}(\widehat{Q}fg) \big| \ = \ C_1 \sup_{\|g\|_q \le 1} \big| \mathrm{tr}(f\widehat{Q}g) \big| \\ &\le C_1 \sup_{\|g\|_q \le 1} \big\| \widehat{Q}g \big\|_{G_q(\mathcal{M})\rtimes \mathrm{G}} \|f\|_{G_p(\mathcal{M})\rtimes \mathrm{G}} \ \le\ C_1 \sqrt{q}\, \|f\|_{G_p(\mathcal{M})\rtimes \mathrm{G}}. \end{aligned}$$

The last inequality follows from the first assertion for q, the contractivity of $\widehat{Q}$ on $L_p^{rc}(E)$ and the continuity of E on $L_{q/2}$. This completes the proof of the result. $\square$

Remark 4.4. Let us observe that, when restricted to $RC_p(\mathcal{M}) \rtimes G$, the conditional norm provides the expected square function. Indeed, for and orthonormal basis (e_k) and $f = \sum_{k,g} (B(e_k) \otimes f_{g,k}) \lambda(g)$ we find

$$\|f\|_{L_p^c(E)} \ = \ \Big\| \Big(\sum_{k,g,g'} \lambda(g)^{-1} f_{g,k}^* f_{g',k} \lambda(g') \Big)^{\frac{1}{2}} \Big\|_p .$$

For the row norm, we prefer to write $f = \sum_{k,g} \lambda(g)(B(e_k) \otimes f'_{g,k})$ for suitable $f'_{g,k}$.

4.2. **Pisier's method and Riesz transforms.** Pisier's approach to the classical Riesz transform on $\mathbb{R}^n$ gives dimension free estimates and is based on a suitable automorphism group. The same tools will be applicable for semigroups of Fourier multipliers on discrete groups. We denote by λ the Lebesgue measure in $\mathbb{R}^n$, γ the normalized gaussian measure and ν the Haar measure on the Bohr compactification. The automorphism semigroup

$$\beta_t f(x,y) = f(x, tx + y)$$

extends to a measure preserving $*$-homomorphism on $L_\infty(\mathbb{R}^n \times \mathbb{R}^n, \gamma \times \lambda)$ and on $L_\infty(\mathbb{R}^n \times \mathbb{R}^n, \lambda \times \nu)$, including the case $n = \infty$. Moreover, if G acts on $\mathbb{R}^n$ then β_t commutes with the diagonal action. The key point in Pisier's argument is to identify the Riesz transform as a combination of the transferred Hilbert transform

$$Hf \ = \ \text{p.v.} \, \frac{1}{\pi} \int_{\mathbb{R}} \beta_t(f) \frac{dt}{t}$$

and the gaussian projection $Q : L_p(\mathbb{R}^n, \gamma) \to \overline{L_p - \text{span}} \big\{ B(\xi) \, | \, \xi \in \mathbb{R}^n \big\}$. Here the gaussian variables are given by $B(\xi)(y) = \langle \xi, y \rangle$, homogenous polynomials of degree 1. Pisier's magic formula for smooth f is given by

$$\sqrt{\frac{2}{\pi}} \, \delta \Delta^{-\frac{1}{2}} f = (Q \otimes id_{\mathbb{R}^n}) \Big(\text{p.v.} \, \frac{1}{\pi} \int_{\mathbb{R}} \beta_t f \frac{dt}{t} \Big),$$

where $\delta : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is given by

$$\delta(f)(x,y) = \sum_{k=1}^n \frac{\partial f}{\partial x_k} y_k = \langle \nabla f, y \rangle.$$

Lemma 4.5. *If $1 < p < \infty$, we find*

$$\delta \Delta^{-\frac{1}{2}} \rtimes id_{\mathrm{G}} : L_p \big(L_\infty(\mathbb{R}^n, \nu) \rtimes \mathrm{G} \big) \to L_p \big(L_\infty(\mathbb{R}^n \times \mathbb{R}^n, \gamma \times \nu) \rtimes \mathrm{G} \big)$$

with norm bounded by $Cp^3/(p-1)^{3/2}$. Moreover, the same holds for $n = \infty$.

Proof. The cross product extension of Pisier's formula reads as

$$\sqrt{\frac{2}{\pi}} \, \big(\delta \Delta^{-\frac{1}{2}} \rtimes id_{\mathrm{G}} \big) f \ = \ \big((Q \otimes id_{\mathbb{R}^n}) \rtimes id_{\mathrm{G}} \big) \Big(\text{p.v.} \, \frac{1}{\pi} \int_{\mathbb{R}} (\beta_t \rtimes id_{\mathrm{G}}) f \frac{dt}{t} \Big).$$

This gives $\delta \Delta^{-\frac{1}{2}} \rtimes id_{\mathrm{G}} \ = \ \sqrt{\pi/2} \, \widehat{Q}(H \rtimes id_{\mathrm{G}})$. By transference, the first mapping is bounded on L_p with constant $\sim p^2/p - 1$. Then, the assertion follows from the complementation result in Theorem 4.3. The proof is complete. $\square$

Given a length function $\psi : \mathrm{G} \to \mathbb{R}_+$, let us consider the gaussian derivation $\delta_\psi : \mathcal{L}(\mathrm{G}) \to L_\infty(\mathbb{R}^n, \gamma) \rtimes \mathrm{G}$ which is determined by $\delta_\psi(\lambda(g)) = B(b_\psi(g))\lambda(g)$, where $b_\psi : \mathrm{G} \to \mathbb{R}^n$ is the (left) cocycle map associated to ψ. Note that we include

the case $n=\infty$, so that any length function/cocycle is included. Let us also consider the generator $A_\psi(\lambda(g)) = \psi(g)\lambda(g)$. Then we use our usual $*$-homomorphism $\pi(\lambda(g)) = \exp b_\psi(g)\lambda(g)$ and observe that $(\delta \rtimes id_{\mathrm{G}}) \circ \pi = (id_{\mathbb{R}^n} \rtimes \pi) \circ \delta_\psi$. This yields the intertwining identity

$$\big(\delta\Delta^{-\frac12} \rtimes id_{\mathrm{G}}\big) \circ \pi \,=\, (id_{\mathbb{R}^n} \rtimes \pi) \circ \delta_\psi A_\psi^{-\frac12}.$$

By Lemma 4.5, both sides are bounded $L_p(\mathcal{L}(\mathrm{G})) \to L_p(L_\infty(\mathbb{R}^n \times \mathbb{R}^n, \gamma \times \nu) \rtimes \mathrm{G})$.

Theorem 4.6. *If $1<p<\infty$, we find*

$$\big\|A_\psi^{\frac12} f\big\|_p \sim_{c_1(p)} \|\delta_\psi f\|_p \sim_{c_2(p)} \|\delta_\psi f\|_{RC_p(\mathbb{C})\rtimes\mathrm{G}}.$$

Moreover, $E(\delta_\psi f^* \delta_\psi f) = \Gamma_\psi(f,f)$ and $E(\delta_\psi f \delta_\psi f^*) = \Gamma_\psi(f^*,f^*)$ so that

$$\big\|A_\psi^{\frac12} f\big\|_p \sim_{c(p)} \begin{cases} \displaystyle\inf_{\delta_\psi f = \phi_1+\phi_2} \big\|E(\phi_1\phi_1^*)\big\|_p + \big\|E(\phi_2^*\phi_2)\big\|_p & \text{if } 1<p\le 2, \\ \max\Big\{ \big\|\Gamma_\psi(f,f)^{\frac12}\big\|_p,\, \big\|\Gamma_\psi(f^*,f^*)^{\frac12}\big\|_p \Big\} & \text{if } 2\le p<\infty. \end{cases}$$

Proof. We have

$$\|\delta_\psi f\|_p = \big\|(id_{\mathbb{R}^n} \rtimes \pi)\delta_\psi A_\psi^{-\frac12}(A_\psi^{\frac12} f)\big\|_p = \big\|(\delta\Delta^{-\frac12} \rtimes id_{\mathrm{G}})\pi(A_\psi^{\frac12} f)\big\|_p \le c_1(p)\big\|A_\psi^{\frac12} f\big\|_p.$$

According to Lemma 4.5, we find $c_1(p) \le p^3/(p-1)^{3/2}$. The upper estimate follows with the same constant from a duality argument. Indeed, given a trigonometric polynomial f, there exists a trigonometric polynomial f' with $\|f'\|_{p'} = 1$ and such that

$$(1-\varepsilon)\big\|A_\psi^{\frac12} f\big\|_p \le \tau_{\mathrm{G}}\big(f' A_\psi^{\frac12} f\big).$$

Note that $A_\psi^{-1/2}$ is only well-defined on $f'' = \sum_{\psi(g)\neq 0} \widehat{f'}(g)\lambda(g)$. However, since $\mathrm{G}_0 = \{g \in \mathrm{G} \,|\, \psi(g) = 0\}$ is a subgroup, we may consider the associated conditional expectation E_{G_0} on $\mathcal{L}(\mathrm{G})$ and obtain $f'' = f' - E_{\mathrm{G}_0} f'$ so that $\|f''\|_{p'} \le 2$. On the other hand, we note the crucial identity

$$\begin{aligned} &\mathrm{tr}_{L_\infty(\Omega)\rtimes\mathrm{G}}\big(\delta_\psi f_1^* \delta_\psi f_2\big) \\ &\quad= \sum_{g_1,g_2} \overline{\widehat{f}_1(g_1)}\widehat{f}_2(g_2)\Big(\int_\Omega \alpha_{g_1^{-1}}\big(\overline{B(b_\psi(g_1))}B(b_\psi(g_2))\big)\,d\mu\Big)\tau_{\mathrm{G}}(\lambda(g_1^{-1}g_2)) \\ &\quad= \sum_{g\in\mathrm{G}} \overline{\widehat{f}_1(g)}\widehat{f}_2(g)\langle b_\psi(g), b_\psi(g)\rangle_\psi \,=\, \sum_{g\in\mathrm{G}} \overline{\widehat{f}_1(g)}\widehat{f}_2(g)\psi(g) \,=\, \tau_{\mathrm{G}}(A_\psi^{\frac12} f_1^* A_\psi^{\frac12} f_2). \end{aligned}$$

Combining both results we get

$$\begin{aligned} \big\|A_\psi^{\frac12} f\big\|_p &\le \frac{1}{1-\varepsilon}\tau_{\mathrm{G}}\big(f' A_\psi^{\frac12} f\big) \,=\, \frac{1}{1-\varepsilon}\tau_{\mathrm{G}}\big(f'' A_\psi^{\frac12} f\big) \\ &= \frac{1}{1-\varepsilon}\,\mathrm{tr}_{L_\infty(\Omega)\rtimes\mathrm{G}}\,\big(\delta_\psi(A_\psi^{-\frac12} f')\delta_\psi f\big) \\ &\le \frac{1}{1-\varepsilon}\big\|\delta_\psi A_\psi^{-\frac12} f'\big\|_{p'}\,\|\delta_\psi f\|_p \,\lesssim\, \frac{p^3}{(p-1)^{\frac32}}\,\|\delta_\psi f\|_p, \end{aligned}$$

where the last estimate follows from Lemma 4.5. This proves our first isomorphism with $c_1(p) \le Cp^3/(p-1)^{3/2}$. Since $\delta_\psi f \in G_p(\mathbb{C}) \rtimes \mathrm{G}$, the second one follows from our Khintchine type inequality and the constants $c_2(p)$ are completely determined by Theorem 4.3. To obtain the last assertion, it suffices to prove the identities

$E(\delta_\psi f^* \delta_\psi f) = \Gamma_\psi(f,f)$ and $E(\delta_\psi f \delta_\psi f^*) = \Gamma_\psi(f^*,f^*)$. Since both are similar, we just prove the first one. Arguing as above, we find

$$E\big(\delta_\psi f^* \delta_\psi f\big) \ = \ \sum_{g,h} \overline{\widehat{f}(g)} \widehat{f}(h) \, \langle b_\psi(g), b_\psi(h) \rangle_\psi \, \lambda(g^{-1}h) \ = \ \Gamma_\psi(f,f)$$

since we know that $\langle b_\psi(g), b_\psi(h) \rangle_\psi = K_\psi(g,h) = \frac{1}{2}(\psi(g) + \psi(h) - \psi(g^{-1}h)$. $\square$

Theorem D in the Introduction follows from the last assertion of Theorem 4.6. When $1 < p < 2$, we may consider decompositions of $f = f_1 + f_2$ so that $\phi_j = \delta_\psi f_j$ in our result. These particular decompositions give rise to

$$\big\| A_\psi^{\frac12} f \big\|_p \, \le \, c(p) \inf_{f = f_1 + f_2} \big\| \Gamma_\psi(f_1, f_1) \big\|_p + \big\| \Gamma_\psi(f_2^*, f_2^*) \big\|_p.$$

We suspect the reverse inequality does not hold. This would require to show that the subspace of derivations of functions is complemented in $G_p(\mathbb{C}) \rtimes \mathrm{G}$ for $1 < p < 2$.

As a byproduct of our methods, we obtain L_p estimates for ψ-directional Riesz transforms associated to arbitrary —not necessarily finite dimensional— cocycles of our discrete group G.

Corollary 4.7. *Given a discrete group* G *and any length function* $\psi : \mathrm{G} \to \mathbb{R}_+$ *with associated* (*not necessarily finite dimensional*) *cocycle* $b_\psi : \mathrm{G} \to \mathcal{H}_\psi$, *let us fix a unit vector* $\eta \in \mathcal{H}_\psi$. *Then, the* ψ*-directional Riesz transform*

$$R_\eta \Big(\sum\nolimits_g \widehat{f}(g) \lambda(g) \Big) \ = \ -i \sum\nolimits_g \frac{\langle b_\psi(g), \eta \rangle_\psi}{\sqrt{\psi(g)}} \widehat{f}(g) \lambda(g)$$

in the direction of η *defines a bounded map* $L_p(\widehat{\mathbb{G}}) \to L_p(\widehat{\mathbb{G}})$ *for all* $1 < p < \infty$.

Proof. We have

$$R_\eta f \ = \ -i E \Big((B(\eta) \mathbf{1}_{\mathcal{L}(\mathrm{G})})^* (\delta_\psi A_\psi^{-\frac12} f) \Big).$$

By self-adjointness, it suffices to consider the case $p \ge 2$ and we find

$$\begin{aligned} \|R_\eta f\|_p & \le \ \big\| E\big(|B(\eta) \mathbf{1}_{\mathcal{L}(\mathrm{G})}|^2\big)^{\frac12} \big\|_{p'} \big\| E\big(|\delta_\psi A_\psi^{-\frac12} f|^2\big)^{\frac12} \big\|_p \\ & = \ \big\| E\big(|\delta_\psi A_\psi^{-\frac12} f|^2\big)^{\frac12} \big\|_p \ \le \ \big\| \delta_\psi A_\psi^{-\frac12} f \big\|_p \ \le \ \frac{Cp^3}{(p-1)^{3/2}} \, \|f\|_p. \end{aligned}$$ $\square$

4.3. **Radial Fourier multipliers.** We now present a transference method between radial Fourier multipliers on discrete groups and their Euclidean counterparts. As usual, given a length function $\psi : \mathrm{G} \to \mathbb{R}_+$, we write $\mathcal{S}_\psi = (S_{\psi,t})_{t \ge 0}$ for the semigroup $\lambda(g) \mapsto \exp(-t\psi(g))\lambda(g)$ and $\mathcal{S} = (S_t)_{t \ge 0}$ for the heat semigroup on $\mathbb{R}^n$ or its Bohr compactification.

Proof of Theorem E. The equivalence i) $\Leftrightarrow$ ii) is clearly a particular case of point D in the proof of Theorem 2.4 since the row case is justified in the exact same manner. The implication iii) $\Rightarrow$ ii) follows by taking $(\mathrm{G}, \psi) = (\mathbb{R}^d_{\mathrm{disc}}, |\ |^2)$, while the argument for i) $\Rightarrow$ iii) is implicit in the proof of the result proved in Paragraph 2.4. Indeed, if b_ψ is the inclusion map associated to either the left or the right cocycle for ψ, we note that

$$m = \widetilde{m} \circ b_\psi \quad \text{for} \quad m = h \circ \psi \quad \text{and} \quad \widetilde{m} = h \circ |\ |^2.$$

This proves the first statement. In fact, the argument for i) $\Rightarrow$ iii) also applies assuming boundedness on the Bohr compactification instead. Moreover, a careful look at this argument shows that all what is needed is Lemma 1.2 for equivariant CZO's and the intertwining identities in Theorem 2.4. Particularly, nothing is affected when we take $d = \infty$ as far as we remove condition i). This shows that ii) $\Leftrightarrow$ iii) even in the infinite-dimensional setting. $\square$

Remark 4.8. Boundedness is equivalent to cb-boundedness for all these maps. Indeed, let cb-j) denote the cb-version of j). Then, the assertion clearly follows from the chain i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) $\Rightarrow$ cb-iii) $\Rightarrow$ cb-ii) $\Rightarrow$ cb-i). The implication cb-iii) $\Rightarrow$ cb-ii) is trivial, while the last implication follows again from the argument for point D in the proof of Theorem 2.4. Therefore, it suffices to show that ii) $\Rightarrow$ cb-iii) which follows again from (the last statement in) Lemma 1.2 and the intertwining identities. This completes the proof.

Remark 4.9. It is standard that

$$\big|\partial_\xi^\beta \widetilde{m}(\xi)\big| \, \le \, c_n |\xi|^{-|\beta|} \quad \Rightarrow \quad \sup_{R>0} \Big(\frac{1}{R^{n-2|\beta|}} \int_{R<|\xi|<2R} \big|\partial_\xi^\beta \widetilde{m}(\xi)\big|^2 \, d\xi \Big)^{\frac12} \, \le \, c_n.$$

If the inequality on the right holds for all $|\beta| \le [\frac{n}{2}] + 1$, we say that $\widetilde{m}$ satisfies Hörmander's smoothness condition. This condition also implies the L_p as well as the $L_\infty \to \mathrm{BMO}$ boundedness of the Fourier multiplier $T_{\widetilde{m}}$ on $\mathbb{R}^n$. Thus, by Theorem E we see that whenever $\widetilde{m}$ satisfies the (weaker) Hörmander smoothness condition and $\widetilde{m} = h \circ |\,|^2$, the Fourier multipliers $T_{h\circ\psi}$ are L_p and $L_\infty \to \mathrm{BMO}$ bounded for any discrete group G with $\dim \mathcal{H}_\psi = n$.

5. Illustrations and comments

We finally illustrate our main results in a variety of scenarios. This will include new examples of L_p bounded Fourier multipliers on $\mathbb{R}^n$ —via a reformulation of Hörmander and de Leeuw theorems in terms of length functions/cocycles— Mihlin type multipliers and dimension free estimates on $\mathbb{R}^n$, noncommutative tori or the free group algebra and new examples of Rieffel's quantum metric spaces. We also add at the end a brief algebraic/geometric analysis of our results.

5.1. **The n-torus.** Since

$$\sum\nolimits_{j,k} \overline{\beta}_j \beta_k \, e^{-t\|k-j\|^2} = \Big(\frac{\pi}{t}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\pi^2\|x\|^2/t} \Big| \sum\nolimits_j \beta_j \, e^{2\pi i \langle j,x\rangle} \Big|^2 \, dx \ge 0,$$

Schoenberg's theorem gives that $\psi(k) = \|k\|^2$ is a length function on $\mathbb{Z}^n$. Being an abelian group, both Gromov products K_ψ^1 and K_ψ^2 coincide, so that there is just one Hilbert space $\mathcal{H}_\psi$ and one inclusion map $b_\psi : \mathrm{G} \to \mathcal{H}_\psi$. In the specific case considered, the inner product takes the form

$$\Big\langle \sum_{j\in\mathbb{Z}^n} a_j \delta_j, \sum_{k\in\mathbb{Z}^n} a_k \delta_k \Big\rangle_\psi = \sum_{j,k\in\mathbb{Z}^n} a_j a_k \, \langle j,k\rangle_{\mathbb{R}^n} = \Big\| \sum_{j\in\mathbb{Z}^n} a_j j \Big\|_{\mathbb{R}^n}^2.$$

According to Lemma 2.1, we have to quotient out the subspace of finitely supported sequences $(a_j)_{j\in\mathbb{Z}^n}$ for which $\sum_j a_j j = 0$. It is easily checked that the resulting quotient is n-dimensional —so that $\mathcal{H}_\psi \simeq \mathbb{R}^n$— and the map $b_\psi : \mathbb{Z}^n \to \mathbb{R}^n$ becomes the canonical inclusion. This cocycle is equipped with the trivial action $\alpha_{\psi,k} = id_{\mathbb{R}^n}$ for all $k \in \mathbb{Z}^n$. In particular, we find that $|\alpha_\psi(\mathbb{Z}^n)| = 1$ in this case

and Corollary E meets exactly the classical Mihlin condition, so that we recover the original formulation of Hörmander-Mihlin theorem for $\mathbb{T}^n$.

5.2. **On de Leeuw's theorems.** Taking $\mathrm{G} = \mathbb{R}^n_{\mathrm{disc}}$ and recalling K. de Leeuw's compactification theorem [9], our assertion in the Introduction for multipliers in $\mathbb{R}^n$ follows at once from Theorem A and the fact that we may take $\varepsilon = 0$ for abelian groups. Let us analyze this result by considering the variety of finite-dimensional cocycles of $\mathbb{R}^n$. This provides a unified approach towards Mihlin and de Leeuw's theorems and L_p multipliers on $\mathbb{R}^n$ which are apparently new. Our $L_\infty \to \mathrm{BMO}$ results via Theorem B also appear to be unknown for some cocycles.

To construct a generic d-dimensional cocycle for $\mathrm{G} = \mathbb{R}^n$, assume that we have $(n,d) = (n_1,d_1) + (n_2,d_2)$ with $n_j = 0$ iff $d_j = 0$ for $j = 1,2$. Consider a triple $\Sigma = (\eta,\pi,\gamma)$ composed by $\eta \in \mathbb{R}^{d_1}$, a representation $\pi : \mathbb{R}^{n_1} \to O(d_1)$ and a group homomorphism $\gamma : \mathbb{R}^{n_2} \to \mathbb{R}^{d_2}$. Then $b_\Sigma(\xi) = b_\Sigma(\xi_1 \oplus \xi_2) = (\pi(\xi_1)\eta - \eta) \oplus \gamma(\xi_2)$ is a cocycle of $\mathbb{R}^n$ associated to $\mathcal{H}_\Sigma = \mathbb{R}^d$ and $\alpha_{\Sigma,\xi} = \pi(\xi_1) \oplus id_{\mathbb{R}^{d_2}}$. In fact, all possible cocycles $\mathbb{R}^n \to \mathbb{R}^d$ break up into an orthogonal sum of an *inner* and a *proper* part (any of which may vanish) as above. The proper part is always associated to the trivial action. This characterization is not hard and it may be folklore. It was already noticed in [7] and a proof can be easily reconstructed from [77, Exercise 4.5]. Here is a list of applications of Corollary F.

1. *Mihlin theorem.* Apply Theorem A ($\varepsilon = 0$) with the trivial cocycle $\mathbb{R}^n \to \mathbb{R}^n$.

2. *de Leeuw's restriction theorem.* In his paper [9], de Leeuw proved that the restriction to $\mathbb{R}^k$ of any sufficiently smooth function $m : \mathbb{R}^n \to \mathbb{C}$ which defines an L_p-bounded Fourier multiplier, is also L_p-bounded. In our setting, de Leeuw's restriction corresponds to take the standard cocycle $\mathbb{R}^k \to \mathbb{R}^n$ given by the inclusion map associated to the trivial action.

3. *de Leeuw's periodization theorem.* Another consequence of de Leeuw's approach is that $\mathbb{Z}^n$-periodizations of L_p-multipliers in $\mathbb{R}^n$ supported by the unit cube remain in the same class, see also Jodeit [22]. This corresponds to $b : \mathbb{R}^n \to \mathbb{R}^{2n}$ given by $b_\Sigma(\xi) = \sum_j (e^{2\pi i \xi_j} - 1)e_j$, with action $\alpha_{\Sigma,\xi}(\zeta) = \sum_j e^{2\pi i \xi_j} \zeta_j e_j$ for $\zeta_j \in \mathbb{C}$. Our $L_\infty \to \mathrm{BMO}$ estimate is apparently new.

4. *Directional multipliers.* Taking $b_\Sigma(\xi) = \sum_j \xi_j \gamma_j$, just $[\frac{d}{2}] + 1 = 1$ derivative is needed for the lifting $\widetilde{m}$ since b_Σ is 1-dimensional with trivial action. Letting $\gamma_j = \delta_{j=j_0}$ we obtain multipliers depending only on the j_0-th coordinate. Taking $\gamma_1, \gamma_2, \ldots, \gamma_n$ to be $\mathbb{Z}$-independent, we obtain injective cocycles and multipliers depending only on the direction $u_\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$.

5. *Directional* BMO *spaces.* Our $L_\infty \to \mathrm{BMO}$ estimates from above seem to be of particular interest. Indeed, taking $\widetilde{m}(\xi) = -i\mathrm{sgn}(\xi)$ and b_Σ as above, it turns out that $m_\xi = \widetilde{m}(b_\Sigma(\xi))$ induces the directional Hilbert transform H_{u_γ} in the direction of u_γ. It is well-known that H_{u_γ} is not $L_\infty \to \mathrm{BMO}$ bounded for the classical BMO space. However, Theorem 2.4 provides the alternative space $\mathrm{BMO}_{u_\gamma} = \mathrm{BMO}_{\mathcal{S}_\psi}$ for $\psi(\xi) = |\langle \xi, u_\gamma \rangle|^2$. Recall that this BMO space interpolates with L_p and thus provides the *right endpoint estimate* for the directional Hilbert transform. Moreover, working with proper d-dimensional cocycles we obtain the obvious generalizations for $1 \le d \le n$.

6. *Donut type multipliers.* We now focus our attention on the cocycle outlined in the Introduction. Namely, let $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha/\beta \in \mathbb{R} \setminus \mathbb{Q}$. Consider the cocycle $\mathbb{R} \to \mathbb{R}^4$ given by

$$\begin{aligned} b_\Sigma(\xi) &= \big(e^{2\pi i \alpha \xi} - 1\big) \oplus \big(e^{2\pi i \beta \xi} - 1\big) \\ &= \big(\cos 2\pi\alpha\xi - 1, \sin 2\pi\alpha\xi, \cos 2\pi\beta\xi - 1, \sin 2\pi\beta\xi\big) \end{aligned}$$

with the action $\alpha_{\Sigma,\xi}(z_1, z_2) = (e^{2\pi i\alpha\xi} z_1, e^{2\pi i\beta\xi} z_2)$ for $z_j \in \mathbb{C}$. Geometrically, we embed $\mathbb{R}$ in a 2-dimensional torus as an *infinite non-periodic helix*. This geodesic flow clearly generalizes by taking cocycles $\mathbb{R}^n \to \mathbb{R}^{2d}$ of the form

$$b_\Sigma(\xi) = \bigoplus_{s=1}^{d} \big(e^{2\pi i \sum_j \xi_j \gamma_j^s} - 1\big).$$

7. Further examples arise from mixed —neither inner nor proper— cocycles.

Remark 5.1. If we work with $\mathbb{Z}^n$ instead of $\mathbb{R}^n_{\mathrm{disc}}$, the same comments apply for Fourier multipliers on the n-torus. Moreover, given a locally compact nondiscrete group G, we may also obtain new results for Fourier multipliers on its reduced von Neumann algebra as far as we have an analog of de Leeuw's compactification theorem for the pair $(\mathrm{G}, \mathrm{G}_{\mathrm{disc}})$, see [55] for more on this topic.

5.3. **Unusual Riesz transforms on $\mathbb{R}^n$.** Let us start with the group $\mathrm{G} = \mathbb{R}_{\mathrm{disc}}$. By subordination, it is easy to see that the function $\psi_\alpha(\xi) = |\xi|^\alpha$ is conditionally negative. As we showed above for the n-torus, the length function $\psi(\xi) = |\xi|^2$ yields the standard cocycle on $\mathbb{R}_{\mathrm{disc}}$. Let us focus on $\psi(\xi) = |\xi|$, which has an interesting gradient form. Indeed, defining the intervals

$$I_\xi = \begin{cases} [0, \xi] & \xi \geq 0, \\ [\xi, 0] & \xi < 0, \end{cases}$$

we may write the associated Gromov form as follows

$$K_\psi(x, y) = \frac{|x| + |y| - |x - y|}{2} = \int_{\mathbb{R}} \chi_{I_x}(s) \chi_{I_y}(s) \, ds.$$

This leads to the cocycle $b_\psi : \mathbb{R}_{\mathrm{disc}} \to L_2(\mathbb{R})$ given by $b_\psi(\xi) = \chi_{I_\xi}$, we refer to Paragraph 5.5 below for details in a similar construction. In order to formulate Meyer's Riesz transform, we denote by $B(I)_{I \subset \mathbb{R}}$ a family of gaussian variables satisfying $\int_{\mathbb{R}} B(I) B(J) d\mu = |I \cap J|$. This can be done combining the standard Brownian motions $B_{t,+} = B(I_t)$ and $B_{t,-} = B(I_{-t})$ or by the gaussian measure space construction on $\mathcal{H}_\psi = L_2(\mathbb{R})$. Meyer's Riesz transform is given by

$$R_\psi(e^{2\pi i \xi \cdot}) \, = \, \delta_\psi A_\psi^{-\frac12}(e^{2\pi i \xi \cdot}) \, = \, |\xi|^{-\frac12} \, B(I_\xi) \, e^{2\pi i \xi \cdot}.$$

The action of $\mathbb{R}_{\mathrm{disc}}$ is given by $\alpha_x(B(b_\psi(y))) = B(I_{x+y}) - B(I_x) = B(\alpha_x(b_\psi(y)))$.

Let us now take $\mathrm{G} = \mathbb{R}^n$. In the Introduction, we mentioned the conditionally negative functions $\psi(\xi) = \int_\Omega |\sum_j \xi_j f_j| d\mu$ given by $f_j \in L_1(\Omega, \mu)$. The boundedness of the imaginary powers on L_p follows from Stein [73]. In this case, the cocycle is given by

$$b_\psi(\xi) \, = \, \chi_{I_\xi^+} + \chi_{I_\xi^-} \in L_2(\Omega \times \mathbb{R}, \mu \times \lambda)$$

where λ denotes Lebesgue measure and

$$\begin{aligned} I_\xi^+ &= \Big\{(\omega,s) \, \big| \, 0 \le s \le \sum_{j=1}^n \xi_j f_j(\omega)\Big\}, \\ I_\xi^- &= \Big\{(\omega,s) \, \big| \, \sum_{j=1}^n \xi_j f_j(\omega) \le s \le 0\Big\}. \end{aligned}$$

As above, we use a Brownian motion $B(E)$ with $E \subset \Omega \times \mathbb{R}$ of finite measure. The Meyer's type Riesz transform associated to our cocycle is now given by the following relation

$$R_\psi(e^{2\pi i \langle \xi, \cdot \rangle}) \, = \, \delta_\psi A_\psi^{-\frac12}(e^{2\pi i \langle \xi, \cdot \rangle}) \, = \, \frac{B(I_\xi^+) + B(I_\xi^-)}{\sqrt{\psi(\xi)}} \, e^{2\pi i \langle \xi, \cdot \rangle}.$$

The ψ-directional Riesz transform is given for $\eta \in L_2(\Omega \times \mathbb{R}, \mu \times \lambda)$ by

$$R_\eta(e^{2\pi i \langle \xi, \cdot \rangle}) \, = \, \frac{1}{\sqrt{\psi(\xi)}} \Big(\int_{I_\xi^+ \cup I_\xi^-} \eta(\omega,s) \, d\mu(\omega) ds \Big) \, e^{2\pi i \langle \xi, \cdot \rangle}.$$

The action α is calculated as in the example above, details are left to the reader.

Corollary 5.2. *If $1 < p < \infty$, we find*

i) *Meyer's type formulation. This yields an embedding*

$$R_\psi = \delta_\psi A_\psi^{-\frac12} : L_p(\widehat{\mathbb{R}}^n_{\mathrm{disc}}) \to G_p(\mathbb{C}) \rtimes_\alpha \mathbb{R}^n_{\mathrm{disc}}.$$

ii) *Cocycle formulation. Given any $\eta \in L_2(\Omega \times \mathbb{R})$, we get the L_p estimate*

$$\begin{aligned} \Big\| \sum_{\xi \in \mathbb{R}^n_{\mathrm{disc}}} \Big(\frac{1}{\sqrt{\psi(\xi)}} \int_{I_\xi^+ \cup I_\xi^-} \eta(\omega,s) \, d\mu(\omega) ds \Big) \, \widehat{f}(\xi) e^{2\pi i \langle \xi, \cdot \rangle} \Big\|_{L_p(\widehat{\mathbb{R}}^n_{\mathrm{disc}})} & \\ \le \; c_p \Big\| \sum_{\xi \in \mathbb{R}^n_{\mathrm{disc}}} \widehat{f}(\xi) e^{2\pi i \langle \xi, \cdot \rangle} \Big\|_{L_p(\widehat{\mathbb{R}}^n_{\mathrm{disc}})}. & \end{aligned}$$

iii) *Unusual Riesz transforms. We use de Leeuw's compactification theorem to replace the Bohr compactification of $\mathbb{R}^n$ by the Euclidean space with its usual topology. This yields the following family of L_p-multipliers indexed by $\eta \in L_2(\Omega \times \mathbb{R})$*

$$\widehat{T_\eta f}(\xi) \, = \, \Big(\frac{1}{\sqrt{\psi(\xi)}} \int_{I_\xi^+ \cup I_\xi^-} \eta(\omega,s) \, d\mu(\omega) ds \Big) \, \widehat{f}(\xi).$$

That is, we have $T_\eta : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ for $1 < p < \infty$ and $\eta \in L_2(\Omega \times \mathbb{R})$.

Point iii) gives Riesz transforms for some nonstandard length functions. Other examples arise from the rich family of infinite-dimensional $\mathbb{R}^n$-cocycles. Taking Ω a one-point probability space, we obtain the multipliers

$$\widehat{T_\eta f}(\xi) \, = \, \Big(\frac{1}{\sqrt{|\xi|}} \int_{I_\xi} \eta(x) \, dx \Big) \widehat{f}(\xi) \, = \, m_\eta(\xi) \widehat{f}(\xi).$$

According to Lebesgue differentiation theorem, Mihlin's condition implies $\sqrt{|\xi|}\eta(\xi)$ to be essentially bounded. We only require $\eta \in L_2(\mathbb{R})$ and it is easy to produce examples m_η such that m_η' has infinite many singularities, outside of the scope of Mihlin's condition. More general examples arise from Corollary 5.2.

5.4. **The noncommutative tori.** We now generalize to noncommutative tori the Hörmander-Mihlin conditions. Given $n \geq 1$ and an $n \times n$ antisymmetric matrix Θ with entries $0 \leq \theta_{ij} < 1$, we define the noncommutative torus with n generators associated to the angle Θ as the von Neumann algebra $\mathcal{A}_\Theta$ generated by n unitaries $u_1, u_2, \ldots, u_n$ satisfying the relations $u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j$. Every element of $\mathcal{A}_\Theta$ can be written as an element in the closure of the span of words of the form $w_k = u_1^{k_1} u_2^{k_2} \cdots u_n^{k_n}$ with $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. Moreover, we equip $\mathcal{A}_\Theta$ with the normalized trace

$$\tau(f) = \tau\Big(\sum_{k \in \mathbb{Z}^n} \widehat{f}(k)\, w_k \Big) = \widehat{f}(0).$$

The classical n-dimensional torus corresponds to $\Theta = 0$, so that $\mathcal{A}_0 = L_\infty(\mathbb{T}^n)$. On the other hand, once we have defined $\mathcal{A}_\Theta$, it is clear what should be the aspect of the heat semigroup for noncommutative tori. Namely

$$S_{\Theta,t}(f) = S_{\Theta,t}\Big(\sum_{k \in \mathbb{Z}^n} \widehat{f}(k)\, w_k \Big) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(k)\, e^{-t|k|^2}\, w_k.$$

We may not apply directly any of our results in Section 2 since $\mathcal{A}_\Theta$ is not the group von Neumann algebra of a discrete group. We will use instead that $\mathcal{A}_\Theta$ embeds in the von Neumann algebra of a discretized Heisenberg group. Given an antisymmetric $n \times n$ matrix Θ with entries $0 \leq \theta_{jk} < 1$, consider the bilinear form $B_\Theta : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{R}$ given by $B_\Theta(\xi, \zeta) = \frac{1}{2} \sum_{j,k=1}^n \theta_{jk} \xi_j \zeta_k = \frac{1}{2} \langle \xi, \Theta \zeta \rangle$. Define the discretized Heisenberg group $\mathrm{H}_\Theta = \mathbb{R} \times \mathbb{Z}^n$ with the product

$$(x, \xi) \cdot (z, \zeta) = \big(x + z + B_\Theta(\xi, \zeta), \xi + \zeta \big).$$

Lemma 5.3. *We have*

$$\mathcal{L}(\mathrm{H}_\Theta) = \int_{\mathbb{R}}^{\oplus} \mathcal{A}_{x\Theta}\, dx.$$

Proof. Let λ denote the left regular representation of H_Θ. Since $(x, 0)$ commutes in H_Θ with every (z, ζ), it turns out that $\lambda(\mathbb{R}, 0)$ lives in the center of the algebra $\mathcal{L}(\mathrm{H}_\Theta)$. Using von Neumann's decomposition theorem for subalgebras of the center

$$\mathcal{L}(\mathrm{H}_\Theta) = \int_{\mathrm{sp}(\lambda(\mathbb{R}))}^{\oplus} \mathcal{M}_x\, dx = \int_{\mathbb{R}}^{\oplus} \mathcal{M}_x\, dx.$$

Given $\xi \in \mathbb{Z}^n$, we set $w_\xi = \lambda(0, \xi)$ and observe that $w_\xi w_\zeta = \lambda(B_\Theta(\xi, \zeta), \xi + \zeta)$ implies $w_\xi w_\zeta = \lambda\big(B_\Theta(\xi, \zeta) - B_\Theta(\zeta, \xi), 0\big) w_\zeta w_\xi$. The w_ξ's are generated by the unitaries $u_j = \lambda(0, e_j)$ which satisfy $u_j u_k = e^{2\pi i \theta_{jk} \cdot} u_k u_j$. Moreover, since $\lambda(\mathbb{R})$ acts on $\mathcal{M}_x$ by scalar multiplication we see that

$$\mathcal{M}_x = \big\langle u_j(x) \, \big| \, 1 \leq j \leq n \big\rangle$$

where the $u_j(x)$'s arise from

$$u_j = \int_{\mathbb{R}}^{\oplus} u_j(x)\, dx \quad \text{and satisfy} \quad u_j(x) u_k(x) = e^{2\pi i \theta_{jk} x} u_k(x) u_j(x).$$

Therefore, we have proved that $\mathcal{M}_x = \mathcal{A}_{x\Theta}$ as expected. $\square$

Corollary 5.4. *Given an angle Θ with n generators, let*

$$T_m : \sum_{k \in \mathbb{Z}^n} \widehat{f}(k)\, w_k \mapsto \sum_{k \in \mathbb{Z}^n} m_k\, \widehat{f}(k)\, w_k$$

be the Fourier multiplier on $\mathcal{A}_\Theta$ associated to $m : \mathbb{Z}^n \to \mathbb{C}$. Let $\widetilde{m} : \mathbb{R}^n \to \mathbb{C}$ be a lifting multiplier for m, so that $\widetilde{m}_{|\mathbb{Z}^n} = m$. Then, if $\widetilde{m}$ satisfies $\widetilde{m} \in \mathcal{C}^{[\frac{n}{2}]+1}(\mathbb{R}^n \setminus \{0\})$ and

$$\big|\partial_\xi^\beta \widetilde{m}(\xi)\big| \, \le \, c_n |\xi|^{-|\beta|} \quad \textit{for all multi-index } \beta \textit{ s.t.} \quad |\beta| \le \Big[\frac{n}{2}\Big] + 1,$$

we find $T_m : L_p(\mathcal{A}_\Theta) \stackrel{cb}{\to} L_p(\mathcal{A}_\Theta)$ for all $1 < p < \infty$ and $T_m : L_\infty(\mathcal{A}_\Theta) \stackrel{cb}{\to} \mathrm{BMO}_{\mathcal{S}_\Theta}$.

Proof. Let us consider the heat semigroup $S_{\Theta,t}(\lambda(x,\xi)) = e^{-t|\xi|^2}\lambda(x,\xi)$ and also the length function $\psi(x,\xi) = |\xi|^2$ in H_Θ. Note that $\mathcal{S}_\Theta = \mathcal{S}_\psi$ in the terminology of Section 2. The length function yields to the non-injective cocycle $\mathrm{H}_\Theta \to \mathbb{R}^n$ given by $b_\psi(x,\xi) = \xi$. The associated action is trivial since

$$\alpha_{\psi,(z,\zeta)}(\xi) = \alpha_{\psi,(z,\zeta)}(b_\psi(0,\xi)) = b_\psi((z,\zeta)\cdot(0,\xi)) - b_\psi(z,\zeta) = \xi.$$

In particular, $|\alpha_\psi(\mathrm{H}_\Theta)| < \infty$ and we know that

$$T_M : \sum_{h \in \mathrm{H}_\Theta} \widehat{f}(h)\lambda(h) \mapsto \sum_{h \in \mathrm{H}_\Theta} M_h \widehat{f}(h)\lambda(h)$$

will be cb-bounded $\mathcal{L}(\mathrm{H}_\Theta) \to \mathrm{BMO}_{\mathcal{S}_\Theta}$ as far as we can find a lifting multiplier $\widetilde{m} \circ b_\psi(h) = M_h$ satisfying the smoothness condition in the statement. Note also that the non-injectivity of the cocycle imposes $M_{(x,\xi)} = M_{(z,\xi)}$ for $x, z \in \mathbb{R}$. However, this is not a restriction for the multiplier m_k in the statement since $b_\psi(x,\cdot)$ is injective for any x. In other words, we use $m_k = M_{(0,k)} = \widetilde{m} \circ b_\psi(0,k) = \widetilde{m}(k)$ as expected. Therefore, since

$$M_{(x,\xi)} \text{ is } x\text{-independent} \ \Rightarrow \ T_M = \int_{\mathbb{R}}^{\oplus} T_{m|_{\mathcal{A}_{x\Theta}}} \, dx,$$

we conclude that

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}} \big\| T_m : \mathcal{A}_{x\Theta} \to \mathrm{BMO}_\Theta \big\|_{cb} < \infty.$$

To show complete boundedness for $x = 1$, we restrict the above inequality to the C^*-algebra generated by the u_j's, where the $\mathcal{A}_{x\Theta}$-norm is x-continuous in the sense of continuous fields [69]. This proves the $L_\infty \to \mathrm{BMO}$ cb-boundedness for $x = 1$ by weak-$*$ density. The $L_p(\mathcal{A}_\Theta) \to L_p(\mathcal{A}_\Theta)$ cb-boundedness is proved as usual by interpolation and duality since the semigroup $\mathcal{S}_\Theta$ is regular. □

Remark 5.5. There is an alternative proof of Corollary 5.4, by a noncommutative form of Calderón's transference method. That way, we can deduce the statement from the classical Hörmander-Mihlin theorem in the n-dimensional torus, see [29] for details. On the other hand, Chen, Xu and Yin have recently extended to $\mathcal{A}_\Theta$ several results from classical harmonic analysis on $\mathbb{T}^n$, see [3]. Also based on a transference argument, they have independently proved that cb-multipliers on the quantum n-torus are exactly those on the usual n-torus with equal cb-norms. We also refer to Neuwirth/Ricard's paper [52] for closely related methods.

With the help of the Heisenberg group H_Θ we can easily obtain Meyer type Riesz transform estimates for the rotation algebras $\mathcal{A}_\Theta$. In fact, we work again with the length function $\psi(x,\xi) = |\xi|^2$. Then, the gaussian random variables are realized on $(\mathbb{R}^n, \gamma)$ with respect to the standard normalized gaussian measure. Hence, since the action is trivial we deduce from Theorem 4.6 that

$$R_\psi = \delta_\psi A_\psi^{-\frac{1}{2}} : L_p(\mathcal{L}(\mathrm{H}_\Theta)) \to L_p\big(\mathbb{R}^n, \gamma; L_p(\mathcal{L}(\mathrm{H}_\Theta))\big).$$

is bounded (even completely bounded) for $1 < p < \infty$. It is easy to see that the direct integral decomposition of Lemma 5.3 amounts to a direct integral of the corresponding L_p space (see [75] for $p = 1$)

$$\|F\|_{L_p(\mathcal{L}(\mathrm{H}_\Theta))} = \int_{\mathbb{R}} \|F(x)\|_{L_p(\mathcal{A}_{x\Theta})} dx.$$

This yields $R_\psi(w_k) = \frac{B(k)}{|k|} w_k$. We also find the same directional derivatives

$$R_\eta(w_k) \, = \, \Big(\frac{1}{|k|} \sum\nolimits_j \eta_j k_j \Big) w_k$$

as for the corresponding commutative tori. Alternatively, we could have derived this result from a direct application of Pisier's method using suitable commuting automorphisms. It is fair to say that —with respect to ψ— the first order differential forms do not depend on the deformation given by Θ.

5.5. **The free group algebra $\mathcal{L}(\mathbb{F}_n)$.** Let $(\mathrm{G}, \psi) = (\mathbb{F}_n, | \ |)$ be the free group on n generators with the standard length function counting the number of letters of a word written in its reduced form. The fact that this function is conditionally negative goes back to Haagerup [18]. When $n = 1$, the group is again abelian and we just have one Hilbert space $\mathcal{H}_\psi$ and a single inclusion map $b_\psi : \mathbb{Z} \to \mathcal{H}_\psi$. The form K_ψ is given by

$$K_\psi(j,k) = \frac{|j| + |k| - |k-j|}{2} = \begin{cases} \min(|j|,|k|) & \text{if } jk > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The system $\xi_j = \delta_j - \delta_{j - \mathrm{sgn}(j)} + N_\psi$ for all $j \in \mathbb{Z} \setminus \{0\}$ is orthonormal as it can be easily checked. Let us see that it generates $\mathcal{H}_\psi$. Indeed, it is obvious that δ_0 belongs to N_ψ and we may write

$$\sum_{j \in \mathbb{Z}} a_j \delta_j = \sum_{j>0} \Big(\sum_{k \ge j} a_k \Big) \xi_j + \sum_{j<0} \Big(\sum_{k \le j} a_k \Big) \xi_j + \Big(\sum_{k \in \mathbb{Z}} a_k \Big) \delta_0.$$

Moreover, this shows that N_ψ is the subspace of $\mathbb{R}[\mathbb{Z}]$ generated by δ_0 and that $\dim \mathcal{H}_\psi = \infty$. In the case of several free generators, the situation is similar. Given two words $g, h \in \mathbb{F}_n$, we have $K_\psi^1(g,h) = |\min(g,h)|$ where $\min(g,h)$ is the longest word inside the common branch of g and h in the Cayley graph. If g and h do not share a branch in the Cayley graph, we let $\min(g,h)$ be the empty word. On the other hand, given a word g, we write g^- for the word which results after deleting the last generator on the right of g. Then, the orthonormal basis of $\mathcal{H}_\psi^1$ is given by $\xi_g = \delta_g - \delta_{g^-} + N_\psi$ for $g \in \mathbb{F}_n \setminus \{e\}$ and N_ψ^1 is generated by δ_e since

$$\sum_{g \in \mathbb{F}_n} a_g \delta_g = \sum_{g \in \mathbb{F}_n} \Big(\sum_{|h| \ge |g|} a_h \Big) \xi_g \quad \text{where we set} \quad \xi_e = \delta_e.$$

This yields the cocycle $b_\psi^1 : g \in \mathbb{F}_n \mapsto \delta_g - \delta_e \in \ell_2(\mathbb{F}_n)/\langle \delta_e \rangle$. The right cocycle only requires slight modifications. Now we may construct Riesz transforms on the free group algebra as follows. Given $f = \sum_g \widehat{f}(g) \lambda(g)$, we find

$$\begin{aligned} \Gamma_\psi(f,f) &= \sum\nolimits_{g,h} \overline{\widehat{f}(g)} \, \widehat{f}(h) \, K_\psi^1(g,h) \, \lambda(g^{-1}h), \\ \Gamma_\psi(f^*,f^*) &= \sum\nolimits_{g,h} \widehat{f}(g) \, \overline{\widehat{f}(h)} \, K_\psi^2(g,h) \, \lambda(gh^{-1}). \end{aligned}$$

On the other hand, given $\eta = \sum_{h \neq e} a_h \delta_h \in \mathcal{H}_\psi^1 = \ell_2(\mathbb{F}_n)/\langle \delta_e \rangle$ we consider

$$\begin{aligned} \mathcal{R}_\eta^1 \Big(\sum\nolimits_g \widehat{f}(g)\lambda(g) \Big) & \\ &= -i \sum\nolimits_g \frac{\big\langle b_\psi^1(g), \eta \big\rangle_\psi}{\sqrt{|g|}} \widehat{f}(g)\lambda(g) \\ &= -i \sum_{g,h} \frac{a_h}{\sqrt{|g|}} \Big(\sum_{\substack{g' \leq g \\ h' \leq h}} \big\langle \xi_{g'}, \xi_{h'} \big\rangle_\psi \Big) \widehat{f}(g)\lambda(g) \\ &= -i \sum_{g,h} \frac{a_h}{\sqrt{|g|}} \big| \min(g,h) \big| \widehat{f}(g)\lambda(g) \ = \ -i \sum_{g,h} \frac{a_h}{\sqrt{|g|}} K_\psi^1(g,h) \widehat{f}(g)\lambda(g), \end{aligned}$$

where the order is determined by the Cayley graph, as in the definition of g^- above.

Corollary 5.6. *We have*

i) *Meyer's formulation for* $2 \leq p < \infty$

$$\Big\| \sum\nolimits_g \sqrt{|g|}\, \widehat{f}(g)\, \lambda(g) \Big\|_p \sim \big\| \Gamma_\psi(f,f)^{\frac12} \big\|_p + \big\| \Gamma_\psi(f^*,f^*)^{\frac12} \big\|_p.$$

ii) *Cocycle formulation for* $1 < p < \infty$

$$\Big\| \sum\nolimits_g \frac{1}{\sqrt{|g|}} \Big(\sum_{h \leq g} a_h \Big) \widehat{f}(g)\lambda(g) \Big\|_p \lesssim \Big\| \sum\nolimits_g \widehat{f}(g)\lambda(g) \Big\|_p.$$

We refer to Paragraph 4.2 *for the modifications on Meyer's form for* $1 < p \leq 2$.

All what is needed to apply Theorems A and B is to know the more we can about finite-dimensional cocycles on $\mathbb{F}_n$. These cocycles are easy to classify. It suffices to know $b(g_k)$ and α_{g_k} for the generators g_k, but any choice of points and unitaries in $\mathbb{R}^d$ is admissible by freeness. Thus, the family of finite-dimensional cocycles of $\mathbb{F}_n$ is too rich. We will concentrate on describing low dimensional injective cocycles since they can be regarded as basic building blocks of our family. Indeed, it is a good exercise to construct higher dimensional cocycles for $\mathbb{F}_n$ out of the ones considered below. Our problem simplifies since $\mathbb{F}_n$ embeds isomorphically into $\mathbb{F}_2$ for all $n \geq 2$. Consider the free group $\mathbb{F}_2$ with two generators a_1, a_2. The construction below is well-known to group/measure theorists. Our first observation goes back to the proof of the Banach-Tarski paradox. Namely, if $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Q}$ the subgroup of $SO(3)$ generated by

$$A_1 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

is isomorphic to $\mathbb{F}_2$ under the mapping

$$\mathbb{F}_2 \ni \underbrace{a_{k_1}^{n_1} a_{k_2}^{n_2} \cdots a_{k_r}^{n_r}}_{w} \mapsto \underbrace{A_{k_1}^{n_1} A_{k_2}^{n_2} \cdots A_{k_r}^{n_r}}_{W_\theta} \in SO(3)$$

with $k_1, k_2, \ldots, k_r \in \{1,2\}$, $k_j \neq k_{j+1}$ and $n_1, n_2, \ldots, n_r \in \mathbb{Z}$. On the other hand $SO(3)$ acts naturally on $\mathbb{R}^3$ and $\alpha_\theta(w) = W_\theta$ defines an isometric action $\mathbb{F}_2 \curvearrowright \mathbb{R}^3$ with associated cocycle map $b_{\theta\xi}(w) = W_\theta(\xi) - \xi$ for some $\xi \in \mathcal{H}_\theta = \mathbb{R}^3$. Therefore

we find a 3-dimensional cocycle $(\mathcal{H}_\theta, b_{\theta\xi}, \alpha_\theta)$ for any $\xi \in \mathbb{R}^3$. In order to pick ξ so that $b_{\theta\xi}$ is injective we must show that

$$A_\theta = \bigcap_{w \in \mathbb{F}_2 \setminus \{e\}} \big\{ \gamma \in \mathbb{R}^3 \, | \, W_\theta(\gamma) \neq \gamma \big\}$$

is nonempty. However, given $w \in \mathbb{F}_2 \setminus \{e\}$, the orthogonal map W_θ is a nonidentity linear map on $\mathbb{R}^n$. In particular, the Lebesgue measure of $\mathbb{R}^3 \setminus A_\theta$ is zero since it is a countable union of linear subspaces with codimension at least 1. This proves that the action α_θ is weakly free with respect to almost every $\xi \in \mathbb{R}^3$ and for all such ξ's we find an injective $b_{\theta\xi} : \mathbb{F}_2 \to \mathbb{R}^3$. Our construction above is not completely constructive since we have not provided a criterium to pick the right ξ's. If e_1, e_2, e_3 denotes the standard basis of $\mathbb{R}^3$, this can be fixed taking $\mathcal{H}_\theta = \mathbb{R}^9$ and

$$\begin{array}{rcl} \alpha_\theta(w) & = & W_\theta \oplus W_\theta \oplus W_\theta, \\ b_\theta(w) & = & \big(W_\theta(e_1) - e_1\big) \oplus \big(W_\theta(e_2) - e_2\big) \oplus \big(W_\theta(e_3) - e_3\big). \end{array}$$

Corollary 5.7. *Given $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Q}$, consider the free group algebra $\mathcal{L}(\mathbb{F}_2)$ equipped with the cocycle $(\mathcal{H}_\theta, b_\theta, \alpha_\theta)$ above. Let ψ_θ denote the associated length function and fix a function $\widetilde{m} \in \mathcal{C}^5(\mathbb{R}^9 \setminus \{0\})$ with*

$$\big| \partial_\xi^\beta \widetilde{m}(\xi) \big| \lesssim \min \Big\{ |\xi|^{-|\beta|+\varepsilon}, |\xi|^{-|\beta|-\varepsilon} \Big\} \quad \textit{for all} \quad |\beta| \leq 11.$$

Then, if $m : \mathbb{F}_2 \to \mathbb{C}$ is of the form $m_w = \widetilde{m} \circ b_\theta(w)$ we find that

$$T_m : \sum\nolimits_w \widehat{f}(w) \lambda(w) \mapsto \sum\nolimits_w m_w \widehat{f}(w) \lambda(w)$$

defines a bounded map on $L_p(\mathcal{L}(\mathbb{F}_2), \tau)$ for $1 < p < \infty$ and $\mathcal{L}(\mathbb{F}_2) \to \mathrm{BMO}_{\mathcal{S}_{\psi_\theta}}$.

Proof. This is a direct application of Theorem A and Remark 2.6. □

Further results follow inspecting the conditions of the other results from Section 2.

Remark 5.8. Given the free group $\mathbb{F}_n = \langle g_1, g_2, \ldots, g_n \rangle$,

$$g_{k_1}^{r_1} g_{k_2}^{r_2} \cdots g_{k_m}^{r_m} \mapsto \sum_{s=1}^n \Big(\sum_{k_j = s} r_j \Big) e_s$$

defines a non-injective $\mathbb{Z}^n$-valued cocycle with respect to the trivial action. It vanishes on a normal subgroup $\mathbb{H}_n$ with $\mathbb{F}_n / \mathbb{H}_n \simeq \mathbb{Z}^n$. Hence, the corresponding semigroup $\mathrm{BMO}_{\mathcal{S}_\psi}$ lives in $\mathcal{L}(\mathbb{F}_n/\mathbb{H}_n) \simeq L_\infty(\mathbb{T}^n)$. Corollary E shows that Fourier multipliers on $\mathbb{F}_n$ which are constant in the cosets of $\mathbb{H}_n$ can be analyzed in terms of the corresponding multiplier in the n-torus. Then we may compose the given cocycle with any other cocycle of $\mathbb{Z}^n$ to obtain cocycles of $\mathbb{F}_n$. That way, our exotic examples for the n-torus can be transferred to produce interesting examples in the free group. In fact, the same observation applies for many *finitely-generated* groups. Indeed, according to Grushko-Neumann theorem any finitely-generated G factorizes as a finite free product of finitely-generated freely indecomposable groups $\mathrm{G}_1 * \mathrm{G}_2 * \cdots * \mathrm{G}_n$. Thus, the same construction applies if all the factors G_j have independent generators g_s, for which all reduced words satisfy for every $1 \leq s \leq n$

$$g_{k_1}^{r_1} g_{k_2}^{r_2} \cdots g_{k_m}^{r_m} = e \quad \Rightarrow \quad \sum_{k_j = s} r_j = 0.$$

5.6. **New quantum metric spaces.** The notion of compact quantum metric space was originally introduced by Rieffel [70, 71]. Let $\mathcal{A}$ be a C*-algebra and $\mathcal{B}$ a unital, dense $*$-subalgebra of $\mathcal{A}$. Let $\|\cdot\|_{\mathrm{lip}}$ be a seminorm on $\mathcal{B}$ vanishing exactly on $\mathbb{C}\mathbf{1}_{\mathcal{A}}$. The triple $(\mathcal{A},\mathcal{B},\|\cdot\|_{\mathrm{lip}})$ is called a *compact quantum metric space* if the metric $\rho(\phi_1,\phi_2)=\sup\{|\phi_1(x)-\phi_2(x)| \mid x\in\mathcal{B} \text{ and } \|x\|_{\mathrm{lip}}\le 1\}$ coincides with the weak-$*$ topology on the state space $S(\mathcal{A})$. This crucial property is hard to verify in general. Ozawa and Rieffel have found an equivalent condition to this property [53, Proposition 1.3], we rewrite it as a lemma.

Lemma 5.9. *If σ is a state on $\mathcal{A}$ and*

$$\Big\{x\in\mathcal{B} \text{ such that } \|x\|_{\mathrm{lip}}\le 1 \text{ and } \sigma(x)=0\Big\}$$

is relatively compact in $\mathcal{A}$, then $(\mathcal{A},\mathcal{B},\|\cdot\|_{\mathrm{lip}})$ is a compact quantum metric space.

Given a length function $\psi:\mathrm{G}\to\mathbb{R}_+$, let $(\mathcal{H}_\psi,\alpha_\psi,b_\psi)$ be either the associated left or right cocycle. We will say that ψ yields a well-separated metric if

$$\Delta_\psi \;=\; \inf_{b_\psi(g)\neq 0}\psi(g) \;=\; \inf_{b_\psi(g)\neq b_\psi(h)}\big\|b_\psi(g)-b_\psi(h)\big\|_{\mathcal{H}_\psi}^2 \;>\; 0.$$

Lemma 5.10. *If $\dim\mathcal{H}_\psi=n$, we have*

$$\begin{aligned}|\mathcal{B}_{R,\psi}| \;=\; \Big|\Big\{b_\psi(g)\;\Big|\; |b_\psi(g)|\le R\Big\}\Big| &\le\; c_n\Big(1+\frac{2R}{\Delta_\psi}\Big)^n,\\ \Big|\Big\{b_\psi(g)\;\Big|(R-1)\le\; |b_\psi(g)|\le R\Big\}\Big| &\le c_n\big((R+\frac{\Delta_\psi}{2})^n-(R-1-\frac{\Delta_\psi}{2})^n\big)\big).\end{aligned}$$

Proof. If $\xi_1\neq\xi_2$ belong to $\mathcal{B}_{R,\psi}$, we have

$$\Big(\xi_1+\frac{\Delta_\psi}{2}\mathrm{B}_n\Big)\cap\Big(\xi_2+\frac{\Delta_\psi}{2}\mathrm{B}_n\Big)=\emptyset,$$

where B_n denotes the Euclidean unit ball in $\mathcal{H}_\psi$. This shows that

$$\Big|\frac{\Delta_\psi}{2}\mathrm{B}_n\Big|\,|\mathcal{B}_{R,\psi}| \;\le\; \Big|\Big(R+\frac{\Delta_\psi}{2}\Big)\mathrm{B}_n\Big| \quad\Rightarrow\quad |\mathcal{B}_{R,\psi}|\;\le\; c_n\Big(1+\frac{2R}{\Delta_\psi}\Big)^n.$$

The second equation is proved similarly for $R-1>|\Delta_\psi/2$. $\square$

Consider now a discrete group G equipped with a length function ψ. We have noticed above that $\mathrm{G}_0=\{g\in\mathrm{G} \mid \psi(g)=0\}$ is a subgroup of G. If we consider the usual semigroup $\mathcal{S}_\psi$ given by $S_{\psi,t}(\lambda(g))=e^{-t\psi(g)}\lambda(g)$, it follows that

$$L_p^\circ(\widehat{\mathbb{G}})=\Big\{f\in L_p(\widehat{\mathbb{G}}) \;\Big|\; \lim_{t\to\infty}S_{\psi,t}f=0\Big\}=\Big\{f\in L_p(\widehat{\mathbb{G}})\;\Big|\;\tau(f)=\widehat{f}(e)=0\Big\}.$$

Lemma 5.11. *If $\dim\mathcal{H}_\psi=n$, $\Delta_\psi>0$, $|\mathrm{G}_0|<\infty$ and*

$$\big|\widetilde{m}(\xi)\big|\;\le\; c_n(1+|\xi|)^{-(n+\varepsilon)} \quad \textit{for some} \quad \varepsilon>0,$$

we find cb-multipliers $T_m:L_1(\widehat{\mathbb{G}})\to L_\infty(\widehat{\mathbb{G}})$ for $m=\widetilde{m}\circ b_\psi$. In particular

$$\big\|S_{\psi,t}:L_1(\widehat{\mathbb{G}})\to L_\infty(\widehat{\mathbb{G}})\big\|_{cb}\le c(n)t^{-n/2}.$$

Proof. Given $f=\sum_g\widehat{f}(g)\lambda(g)\in S_1^r(L_1(\widehat{\mathbb{G}}))$ with $\widehat{f}(g)\in M_r$, we have

$$\|\widehat{f}(g)\lambda(g)\|_{S_1^r(L_1(\widehat{\mathbb{G}}))}\ge\|\widehat{f}(g)\|_{S_1^r}\ge\|\widehat{f}(g)\lambda(g)\|_{S_1^r(L_\infty(\widehat{\mathbb{G}}))}.$$

Now it suffices to apply the triangle inequality and a counting argument. Since $\mathrm{vol}((k+1)B_2^n \setminus kB_2^n) = (k+1)^n \mathrm{vol}(B_2^n) - k^n \mathrm{vol}(B_2^n)$, we deduce that

$$\begin{aligned}
&\Big\| \sum\nolimits_g m_g \widehat{f}(g)\lambda(g) \Big\|_{S_1^r(L_\infty(\widehat{\mathbb{G}}))} \\
&\quad \le \ |m_e||\mathrm{G}_0| \big\| f(e)\lambda(e) \big\|_{S_1^r(L_1(\widehat{\mathbb{G}}))} \\
&\quad + \ (4\Delta_\psi^{-1})^n \|f\|_{S_1^r(L_1(\widehat{\mathbb{G}}))} \Big(\sup_{\Delta_\psi \le |\xi| \le 2} |m(\xi)| + nC(n,\varepsilon) \sum_{k \ge 2} k^{n-1}(k-1)^{-(n+\varepsilon)} \Big)
\end{aligned}$$

which is dominated by $\|f\|_{S_1^r(L_1(\widehat{\mathbb{G}}))}$. The second assertion follows from

$$\sum_{k \ge 1} e^{-tk^2} k^{n-1} \le C(n)\Gamma(n/2)t^{-n/2}. \qquad \square$$

To state our next result, we need to consider the *gradient form* associated to the infinitesimal generator $A_\psi(\lambda(g)) = \psi(g)\lambda(g)$ of our semigroup $\mathcal{S}_\psi$. Namely, if $\mathbb{C}[\mathrm{G}]$ stands for the algebra of trigonometric polynomials (whose norm closure is the reduced C*-algebra of G), we set for $f_1, f_2 \in \mathbb{C}[\mathrm{G}]$

$$2\Gamma(f_1,f_2) = A_\psi(f_1^*)f_2 + f_1^* A_\psi(f_2) - A_\psi(f_1^* f_2).$$

Consider the seminorm

$$\|f\|_\Gamma = \max \Big\{ \big\|\Gamma(f,f)\big\|_\infty^{\frac12}, \big\|\Gamma(f^*,f^*)\big\|_\infty^{\frac12} \Big\}$$

and the pseudo-metric $\mathrm{dist}_\psi(g,h) = \sqrt{\psi(g^{-1}h)}$. We find the following result.

Corollary 5.12. *If* $\dim \mathcal{H}_\psi < \infty$*, we deduce*

$$\mathrm{dist}_\psi \ \textit{well-separated metric} \ \Rightarrow \ (\mathrm{C}^*_{\mathrm{red}}(\mathrm{G}), \mathbb{C}[\mathrm{G}], \|\cdot\|_\Gamma) \ \textit{quantum metric space.}$$

Proof. Since $\mathrm{dist}_\psi(g,h) = \|b_\psi(g) - b_\psi(h)\|_{\mathcal{H}_\psi}$, it defines a metric iff $b_\psi : \mathrm{G} \to \mathcal{H}_\psi$ is injective iff $\mathrm{G}_0 = \{e\}$. On the other hand, recalling that $\Gamma(f,f) \ge 0$, we see that $\Gamma(f,f) = 0$ iff $\tau(\Gamma(f,f)) = 0$. It is easily checked that $\tau(\Gamma(f,f)) = \sum_g |\widehat{f}(g)|^2 \psi(g)$. Hence we deduce that $\|\cdot\|_\Gamma$ vanishes in $\mathbb{C}\mathbf{1}$ iff $\mathrm{G}_0 = \{e\}$ iff dist_ψ is a metric. It is also clear that dist_ψ is well-separated iff $\Delta_\psi > 0$. In particular, we can not have infinitely many points of $b_\psi(\mathrm{G})$ inside any ball of the finite-dimensional Hilbert space $\mathcal{H}_\psi$. This means that the set $\{\psi(g)^{-1} \,|\, g \neq e\}$ can not have a cluster point different from 0, so that

$$A_\psi^{-1} : L_2^\circ(\widehat{\mathbb{G}}) \to L_2^\circ(\widehat{\mathbb{G}})$$

is a compact operator. According to [25, Theorem 1.1.7], $A_\psi^{-1/2} : L_p^\circ(\widehat{\mathbb{G}}) \to L_\infty^\circ(\widehat{\mathbb{G}})$ is also compact for any $p > n + \varepsilon$. Lemma 5.11 has been essential at this point, see [25]. This means that

$$\Big\{ f \in L_\infty^\circ(\widehat{\mathbb{G}}) \,\big|\, \big\|A_\psi^{1/2} f\big\|_p \le 1 \Big\} = \Big\{ f \in L_\infty(\widehat{\mathbb{G}}) \,\big|\, \big\|A_\psi^{1/2} f\big\|_p \le 1 \text{ and } \tau(f) = 0 \Big\}$$

is relatively compact in $L_\infty(\widehat{\mathbb{G}})$. According to the main result in [25], we see that

$$\big\|A_\psi^{1/2} f\big\|_p \ \le \ c_p \max \Big\{ \big\|\Gamma(f,f)^{\frac12}\big\|_p, \big\|\Gamma(f^*,f^*)^{\frac12}\big\|_p \Big\} \ \le \ c_p \|f\|_\Gamma.$$

We deduce from this inequality that

$$\Big\{ f \in L_\infty(\widehat{\mathbb{G}}) \,\big|\, \|f\|_\Gamma \le 1 \text{ and } \tau(f) = 0 \Big\}$$

is relatively compact in $L_\infty(\widehat{\mathrm{G}})$. The desired result follows from Lemma 5.9. □

Remark 5.13. Let $\mathrm{G} = \mathbb{Z}$ and $\psi(k) = |k|^2$. Consider the commutator $[A_\psi^\alpha, f]$ of A_ψ^α and $f \in \mathbb{C}[\mathbb{Z}]$. Rieffel [71] showed that the triple $(\mathrm{C}^*_{\mathrm{red}}(\mathbb{Z}), \mathbb{C}[\mathbb{Z}], \|[A_\psi^\alpha, \cdot\,]\|)$ is a compact quantum metric space for all $0 < \alpha \leq \frac{1}{2}$. The same argument of the previous corollary shows that this is true for $\frac{1}{2} < \alpha \leq 1$ too. Indeed, in this case $n = 1$ and applying Lemma 5.11 together with [25, Theorem 1.1.7], we have that $A_\psi^{-\alpha}$ is compact from $L_2^\circ(\mathbb{T})$ to $L_\infty^\circ(\mathbb{T})$ since we may choose $\varepsilon > 0$ such that $2 > \frac{1+\varepsilon}{2\alpha}$ for any $\alpha > \frac{1}{4}$. In particular, $\{x \in L_\infty^\circ(\mathbb{T}) \,|\, \|A_\psi^\alpha f\|_2 \leq 1\}$ is relatively compact in $L_\infty(\mathbb{T})$. Note that $\|[A_\psi^\alpha, f]\| \geq \|[A_\psi^\alpha, f]\mathbf{1}\|_2 = \|A_\psi^\alpha(f)\|_2$. We conclude that

$$\Big\{ f \in L_\infty(\mathbb{T}) \,\Big|\, \big\|[A_\psi^\alpha, f]\big\| \leq 1 \text{ and } \int_{\mathbb{T}} f \, d\mu = 0 \Big\}$$

is relatively compact. Again, we deduce the assertion from Lemma 5.9. Moreover, the exact same argument applies on $\mathbb{Z}^2$ with $\frac{1}{2} < \alpha \leq 1$ and on $\mathbb{Z}^3$ with $\frac{3}{4} < \alpha \leq 1$.

5.7. **Conclusions.** The classical form of the Hörmander-Mihlin theorem on the compact dual of $\mathbb{Z}^n$ is applied either for *testing* the boundedness of a given multiplier or for *constructing* multipliers out of smooth lifting functions. In both situations the standard length function $\psi(k) = |k|^2$ with its associated cocycle $b_\psi : \mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ are used in conjunction with transference. The properties which characterize this cocycle are $\Delta_\psi > 0$ and the injectivity of the cocycle map b_ψ. The injectivity avoids additional restrictions on the multiplier m under the lifting $m = \widetilde{m} \circ b_\psi$, while the well-separatedness $\Delta_\psi > 0$ preserves the discrete topology of $\mathbb{Z}^n$ in its image on $\mathbb{R}^n$. Here is a description of those finite-dimensional cocycles of $\mathbb{Z}^n$.

Lemma 5.14. *Let ψ be a length function on $\mathbb{Z}^n$ giving rise to a finite-dimensional cocycle $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$. Assume that* $\dim \mathcal{H}_\psi = d$*, b_ψ is injective and* $\Delta_\psi > 0$*. Then we find that*

- $\alpha_\psi : \mathbb{Z}^n \to \mathrm{Aut}(\mathcal{H}_\psi)$ *is the trivial action,*
- $(\mathcal{H}_\psi, d) \simeq (\mathbb{R}^n, n)$ *and $b_\psi : \mathbb{Z}^n \to \mathcal{H}_\psi$ is a group homomorphism.*

Proof. We know from Paragraph 5.2 that

$$b_\psi(k) \, = \, b_\psi(k_1 \oplus k_2) \, = \, \big(\pi(k_1)\eta - \eta\big) \oplus \gamma(k_2),$$

where $(n,d) = (n_1, d_1) + (n_2, d_2)$, the map $\pi : \mathbb{Z}^{n_1} \to O(d_1)$ is an orthogonal representation, $\gamma : \mathbb{Z}^{n_2} \to \mathbb{R}^{d_2}$ is a group homomorphism, $\eta \in \mathbb{R}^{d_1}$ and the action has the form $\alpha_\psi(k) = \pi(k_1) \oplus id_{\mathbb{R}^{d_2}}$. Moreover, we claim that $(n_1, d_1) = (0,0)$ from the hypotheses. This implies the assertion. Indeed, the action α_ψ must be trivial if b_ψ has no inner part and we get that $b_\psi = \gamma$ is a group homomorphism. We also know that $d \geq n$ from the injectivity of b_ψ and $d \leq n$ since $b_\psi(e_1), b_\psi(e_2), \ldots, b_\psi(e_n)$ linearly generate $\mathcal{H}_\psi$. To prove the claim we assume that $n_1, d_1 > 0$. Then, the injectivity of b_ψ and the condition $\Delta_\psi > 0$ imply that $\{\pi(k_1)\eta - \eta \,|\, k_1 \in \mathbb{Z}^{n_1}\}$ is an infinite set of points in $\mathbb{R}^{d_1}$ mutually separated by a distance greater or equal than $\sqrt{\Delta_\psi} > 0$. This means that the set must be unbounded, which is a contradiction since $\|\pi(k_1)\eta - \eta\|_{\mathcal{H}_\psi} \leq 2\|\eta\|_{\mathcal{H}_\psi}$ for all $k_1 \in \mathbb{Z}^{n_1}$. □

In particular, given a length function $\psi : \mathrm{G} \to \mathbb{R}_+$, it is natural to call ψ a *standard length function* if $\Delta_\psi > 0$ and the associated cocycle $b_\psi : \mathrm{G} \to \mathcal{H}_\psi$ is injective. Although standard cocycles are an important piece of the theory, they

are definitely not the whole of it! We have already illustrated this with our "donut multipliers" above. It turns out that geometrical intuition is not enough to describe arbitrary L_p bounded Fourier multipliers. Let us analyze what new information can be extracted from our results so far.

5.7.1. *Small dimension vs smooth interpolation.* If we are given a fixed multiplier on $\mathcal{L}(\mathrm{G})$, the problem of finding the optimal cocycle and lifting multiplier to study its L_p boundedness might be quite hard.

Problem 5.15. *Given a Fourier multiplier*

$$\sum_g \widehat{f}(g)\lambda(g) \mapsto \sum_g m_g \widehat{f}(g)\lambda(g)$$

a) *Find low dimensional injective cocycles $b_\psi : \mathrm{G} \to \mathcal{H}_\psi$.*
b) *Given such $(\mathcal{H}_\psi, b_\psi, \alpha_\psi)$, find $\widetilde{m} \in \mathcal{C}^{d_n}(\mathbb{R}^n \setminus \{0\})$ with $\widetilde{m}(b_\psi(g)) = m_g$ and minimizing*

$$\sup_{\xi \in \mathbb{R}^n} \sup_{|\beta| \le d_n} |\xi|^{|\beta|+\varepsilon} \big| \partial_\xi^\beta \widetilde{m}(\xi) \big|,$$

where the values of $[\frac{n}{2}] + 1 \le d_n \le n+2$ and $\varepsilon \ge 0$ depend on ψ and m.

Once fixed a cocycle, we must find a lifting multiplier for $m : \mathrm{G} \to \mathbb{C}$ optimizing the constants. This means that we have to control a number of derivatives (see Paragraph 2.4 for the cases where $\varepsilon = 0$ suffices) of a smooth function $\widetilde{m}$ taking certain preassigned values on a cloud of points $b_\psi(g)$ in $\mathbb{R}^n$. In particular, this fits in Fefferman's approach to the *smooth interpolation of data* carried out in [13, 14, 15] and the references therein. There is no canonical answer for questions a) and b) and in general we find certain incompatibility. Indeed, if we pick $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{R}$ linearly independent over $\mathbb{Z}$, the cocycle $k \mapsto \sum_j \gamma_j k_j$ associated to the trivial action is a 1-dimensional injective cocycle for $\mathbb{Z}^n$. This minimizes the number of derivatives to estimate for the lifting multiplier. Note however that $\{\sum_j \gamma_j k_j \mid k \in \mathbb{Z}^n\}$ is a dense cloud of points in $\mathbb{R}$ and the ψ-metric is far to be well-separated. In general this makes harder to solve b), and the lifting multiplier will be highly oscillating in many cases. On the other hand, as we have seen for $\mathbb{R}^n$, certain multipliers can only be treated with alternative cocycles like this one. In summary, our notion of "smooth multiplier" is very much affected by the cocycle we use.

Problem 5.16. *Solve Problem* 5.15 *using standard cocycles, not just injective ones.*

This is more restrictive and we will not always find finite-dimensional standard cocycles, see the next paragraph. On the other hand, if we content ourselves with not necessarily injective well-separated cocycles, we may apply our construction in Remark 5.8 for finitely generated groups.

Problem 5.17. *We do not expect Theorem* A *to be valid with $\varepsilon = 0$ in general due to the missing $L_\infty \to \mathrm{BMO}_r$ estimate, see* [54, Section 6.1] *for counter examples in this line. In other words, this suggests that one could find discrete groups* G *and cocycles $b_\psi : \mathrm{G} \to \mathbb{R}^n$ for which Mihlin regularity $|\partial_\xi^\beta \widetilde{m}(\xi)| \lesssim |\xi|^{-|\beta|}$ for $|\beta| \le [\frac{n}{2}]+1$ is not enough to ensure L_p boundedness of the multipliers $m_g = \widetilde{m}(b_\psi(g))$. Find examples of such bad-behaved cocycles.*

5.7.2. *Infinite-dimensional standard cocycles.* There are two distinguished length functions on $\mathbb{Z}$, the absolute value $\psi(k) = |k|$ and its square respectively related to the Poisson and heat semigroups. Both yield standard cocycles, but one of them is infinite-dimensional while the other has dimension 1. If we take free products of $\mathbb{Z}$ only the Poisson like cocycle survives. Hence we wonder if there exist finite dimensional standard cocycles for the free group. A negative answer follows from a classical theorem of Bieberbach [2]. Let us recall that a group G is called *virtually abelian* whenever it has an abelian subgroup H of finite index, so that G has finitely many left/right H-cosets. Bieberbach's theorem claims that *every discrete subgroup of* $\mathbb{R}^n \rtimes O(n)$ *is virtually abelian.*

Theorem 5.18. *If* G *has a finite-dimensional standard cocycle,* G *is virtually abelian.*

Proof. Note that $g \mapsto (b_\psi(g), \alpha_{\psi,g}) \in \mathcal{H}_\psi \rtimes O(\dim \mathcal{H}_\psi)$ defines an injective group isomorphism for any standard cocycle. Moreover, the well-separatedness property shows that it is an homeomorphism. Thus, G can be regarded as a discrete subgroup of $\mathcal{H}_\psi \rtimes O(\dim \mathcal{H}_\psi)$ with $\dim \mathcal{H}_\psi < \infty$. We conclude from Bieberbach's theorem. □

According to this result, we see in particular that nonabelian free groups do not admit finite-dimensional standard cocycles. A unitary representation of a locally compact group G is called *primary* if the center of its intertwining algebra $\mathcal{C}(\pi)$ is trivial. The group G is said to be of *type* I whenever the von Neumann algebra $\mathcal{A}_\pi$ generated by every primary representation π is a type I factor. This condition turns out to be crucial to admit Plancherel type theorems in terms of irreducible unitary representations, see [16, Chapter 7] for explicit results.

Corollary 5.19. *A discrete group is virtually abelian if and only if it is of type* I.

Proof. By Thoma's theorem [76], a discrete group is type I iff it has a normal abelian subgroup of finite index, hence virtually abelian. On the contrary, if G is virtually abelian it admits an abelian subgroup H of finite index. Let us show that we can pick another such H being a normal subgroup. The map $\gamma : g \mapsto \Lambda_g$ with $\Lambda_g(g'\mathrm{H}) = gg'\mathrm{H}$ defines a group homomorphism between G and the symmetric group of permutations $\mathcal{S}_{\mathrm{G}/\mathrm{H}}$ on the space of left H-cosets. Its kernel is clearly a normal subgroup of G, which is abelian since it is contained in H and of finite index since $\mathrm{G}/\ker\gamma \simeq \mathrm{Img}\,\gamma$ is a subgroup of a finite group, hence finite. □

A locally compact group G satisfies Kazhdan's property (T) when the trivial representation is an isolated point in the dual object with the Fell topology. A discrete group G satisfies this property iff all its cocycles are inner. Moreover, a cocycle is inner iff it is bounded. Hence, Kazhdan property (T) is incompatible with finite-dimensional standard cocycles. In summary, many interesting discrete groups do not admit a *finite-dimensional "standard" Hörmander-Mihlin theory* as it happens with the integer lattice $\mathbb{Z}^n$. Our results establish a more general theory which includes these cases.

5.7.3. *On the Bohr compactification.* We have

$$\int_{\widehat{\mathbb{R}}^n_{\mathrm{disc}}} \lambda(\xi)\, d\mu = \int_{\widehat{\mathbb{R}}^n_{\mathrm{disc}}} e^{2\pi i\langle \xi, x\rangle}\, d\mu(x) = \delta_{\xi,0}$$

for the Haar measure μ on $\widehat{\mathbb{R}}^n_{\mathrm{disc}}$. Being a Haar measure on a compact group, it is a translation invariant probability measure on the Bohr compactification. Therefore it vanishes on every measurable bounded set of $\mathbb{R}^n$ and μ is singular to the Lebesgue measure. In fact

$$\delta_{\xi,0} = \lim_{t\to\infty} \exp(-t|\xi|^2) = \lim_{t\to\infty} \Big(\frac{\pi}{t}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{2\pi i\langle \xi,x\rangle} \exp\Big(-\frac{\pi^2|x|^2}{t}\Big)\,dx,$$

so that we find

$$\int_{\widehat{\mathbb{R}}^n_{\mathrm{disc}}} f\,d\mu = \lim_{t\to\infty} \Big(\frac{\pi}{t}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(x) \exp\Big(-\frac{\pi^2|x|^2}{t}\Big)\,dx.$$

In other words, the measure μ can be understood as a limit of averages along large spheres. By subordination, the same holds for Poisson kernels. As it follows from Theorem E, a dimension-free Calderón-Zygmund theory for Fourier multipliers on arbitrary discrete groups would follow from a dimension-free CZ theory on the Bohr compactification. This lead us to the following very natural problem.

Problem 5.20. *Develop a CZ theory for the heat/Poisson semigroups on* $\widehat{\mathbb{R}}^\infty_{\mathrm{disc}}$.

In order to bring some hope to the problem suggested above, we can construct non-trivial radial Fourier multipliers in the Bohr compactification of $\mathbb{R}^\infty$. In terms of Theorem E, we may equivalently say that the class of radial Fourier multipliers which are bounded $T_{h\circ\psi} : \mathcal{L}(\mathrm{G}) \to \mathrm{BMO}_{\mathcal{S}_\psi}(\mathcal{L}(\mathrm{G}))$ for any discrete group G with $\dim \mathcal{H}_\psi = \infty$ is not trivial. Indeed, as it follows from [26], imaginary powers of length functions are bounded with free-dimensional constants. More concretely given any discrete group G and any length function $\psi : \mathrm{G} \to \mathbb{R}_+$, the family of functions of the form

$$m_g = \sqrt{\psi(g)} \int_{\mathbb{R}_+} e^{-s\sqrt{\psi(g)}} f(s)\,ds$$

with $f : \mathbb{R}_+ \to \mathbb{C}$ bounded, define radial multipliers for which

$$T_m : \mathcal{L}(\mathrm{G}) \to \mathrm{BMO}_{\mathcal{S}_\psi}$$

is bounded and its norm does not depend on $\dim \mathcal{H}_\psi$. Recall that

$$f(s) = \frac{s^{-2i\gamma}}{\Gamma(1-i\gamma)} \quad \text{with} \quad \gamma\in\mathbb{R} \quad \Rightarrow \quad m_g = \psi(g)^{i\gamma}.$$

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